OPTION PRICING IN PERIODS OF NEGATIVE OR LOW INTEREST RATES: CASE OF EUROPEAN TYPE OF OPTIONS

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MASTER THESIS

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ÖZET

NEGATİF VEYA DÜŞÜK FAİZ ORTAMLARINDA OPSİYON FİYATLAMASI: AVRUPA TİPİ OPSİYON UYGULAMASI

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Ekonominin durgunluktan çıkmasına yardımcı olmak için uygulanan negatif faiz politikası, piyasada bazı fiyatlama modellerinin revize edilmesini zorunlu hale getirmektedir. Faiz oranlarının sıfırın altında (negatif) olduğu mevcut durumda Black Scholes modeli, formülünde sadece pozitif oranlara izin verdiği için faiz oranları seçeneklerini fiyatlamamaktadır.

Deloitte, Şubat 2016'da Black Scholes ve SABR modelleri gibi mevcut fiyatlandırma modellerine değişen bir parametre eklemeye dayalı yeni bir fiyatlandırma rejimi uygulamaya koymuştur. Bu çalışmada, negatif faiz oranı ortamında Black Scholes Modelinin performansını artırmak için bir modifiye parametresi atanmıştır. Ayrıca, piyasada halihazırda kullanılan bazı fiyatlandırma tekniklerinin yanı sıra, bu çalışma, hız, karmaşıklık ve doğruluk bazında fiyatlandırma modellerinin karşılaştırmasını da sunmaktadır.

Doğru modifiye parametresini bulmak ve zımni oynaklığı oluşturmak zor olsa da, modifiye Black Scholes modelinin oranlar sıfırın altındayken, özellikle Montecarlo simülasyonu ile desteklendiğinde, faiz oranı seçeneklerini fiyatlandırmak için yeterli olduğu sonucuna ulaşılmıştır.

Anahtar kelimeler: Modifiye Black Scholes modeli, Opsiyon fiyatlaması, Negatif faiz oranı, Zımni oynaklık, Stokastik oynaklık, Faiz ürünleri

ABSTRACT

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The negative interest rates policy that was implemented to help the economy to recover from a recession, pushed the market to revise some pricing models. The Black Scholes model in the current situation where rates are below zero fails to price interest rate options since it only allows positive rates in its formula. Besides the Black Scholes model, the Heston Cox Ingersoll also doesn't allow negative inputs of interest rates.

Deloitte in February 2016 introduced a new pricing regime based on inserting a shifting parameter to existing pricing models like the Black Scholes and the SABR models. This work, analyses the performance of the shifted Black Scholes model in the negative rate environment. In this study, we also conduct a comparative study of pricing models, by providing their performance based on speed, complexity, and market applicability.

According to empirical findings, the shifted Black Scholes model performs very well in the negative rate environment. Even though finding the right shift parameter and generating the implied volatility can be challenging, the shifted Black Scholes, especially when backed with the Montecarlo simulation, is equipped with enough tools to price interest rate options when rates are below zero.

Keywords: Shifted Black model, Option pricing, Negative rates, Implied volatility, Stochastic volatility, Interest rates products**.**

01/07/2022

STATEMENT OF COMPLIANCE WITH ETHICAL PRINCIPLES AND RULES

I hereby truthfully declare that this thesis is an original work prepared by me; that I have behaved in accordance with the scientific ethical principles and rules throughout the stages of preparation, data collection, analysis and presentation of my work; that I have cited the sources of all the data and information that could be obtained within the scope of this study, and included these sources in the references section; and that this study has been scanned for plagiarism with "scientific plagiarism detection program" used by Anadolu University, and that "it does not have any plagiarism" whatsoever. I also declare that, if a case contrary to my declaration is detected in my work at any time, I hereby express my consent to all the ethical and legal consequences that are involved.

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Armand-Charles NGABIRANO

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TABLE OF CONTENTS

LISTE OF TABLES AND FIGURES

1. INTRODUCTION

The recent financial crisis in 2008 revealed the trustworthiness between counterparties as a major concern in financial transactions. The downfall of some financial institutions affected the World's economy. For many, trading became complex or hardly affordable (underpricing, overpricing, credit risk). In order to avoid this chaotic situation bringing a halt to the economy, central banks, especially in Europe, implemented some exceptional measures. Interest rates were lowered in order to make borrowing cheaper (Deloitte, 2016). The idea is to invite investors to borrow money and inject it into the economy. On June 5th, 2014, this process was taken even further, and negative interest rates were observed for the first time in history. Technically a negative rate implies that putting money in a deposit account will result in a loss. Central banks, as a result, punish investors and central banks for holding their cash. This new policy was supposed to inspire investors to bring in new money, which might as a result, contribute to long-term economic growth.

Using negative interest rates has major economic consequences and requires considerable technical implications. Models that help fix prices for derivatives usually require (high) positive risk-free rates, which therefore becomes challenging in the negative rates environment. For example, the cox Ingersoll ross short rate model only works with positive inputs and, as a result, can not be used for negative interest rates. The famous Black Scholes model used primarily in pricing European options strongly works under the assumption that rates can never be negative. The question remains: What derivative pricing model should be used for accurate results in periods of low or negative rates?

Options theory has a long and rich history. The elaboration of the Black Scholes model as a tool of option pricing in 1973 made options trading much more practical. Since that year, traders have widely used the Black Scholes model to price with accuracy options. With the implementation of negative rates, specific models like the Hull and White, the Heston Vasicek, and the Bachelier are regaining their popularity again.

This study will mainly insist on some existing pricing models. As pricing derivatives require a strong understanding of stochastic calculus, the first part reviews the dynamics behind options pricing. From the simple understanding of the Brownian Motion theory to the concept of option pricing under zero-coupon bonds, the first part of this study will provide the necessary pieces of information used in derivative pricing. Along with some python simulations, real-life market examples will be calibrated to get a strong understanding of the theoretical part of this study.

A shift parameter is added to Black's formula for the main problem of pricing under negative rate dynamics. By assuming that the LIBOR rate follows a log-normal process, the concept of shifting a distribution is very similar to the idea of displacement. The process allows moving the probability density along the X-axis to cover the negative territory.

1.1. Research problem

When the negative rate policy was implemented, its primary purpose was to boost the economy recovering from a financial crisis. Even though it seems to be a good policy, it has some drawbacks. When pricing any asset, the risk-free rate must be a reference rate for the asset's return. As a result, most pricing formulas have to incorporate the interest rate part in their dynamics.

When rates become negative, one can imagine how existing pricing models are affected by such change. The idea of having day rate cuts below zero was considered inconceivable. In the world of derivatives, only the Bachelier model under its dynamics assumed that any underlying follows a normal distribution and could evaluate any option regardless of whether the underlying prices are positive or negative. However, as this study will later show, the Bachelier model has its limits.

It is in this perspective that some existing pricing models will be reviewed. At the end of this study, an appropriate, quick and handy model to use in the low rates environment will be determined.

1.2. Objectives and Limitations

The primary purpose is to analyze some of the models commonly used on the market to develop a convenient model that can efficiently deal with negative interest rates and give accurate options prices. Since there are a lot of available models on the market, the main parameters to consider are speed, complexity, and accuracy.

This study will only be limited to interest rate options. European options will be the center of attention since the lowest level of the LIBOR rate is only observed in Europe. This study is conducted during two different periods. First, in 3 months, we price options under the lowest point, then proceed the same for the highest point with a three-month Euribor rate.

1.3. Research Questions

This research addresses the following question:

-Do negative interest rates impact pricing techniques?

-Can existing models overcome the challenge imposed by negative interest rates?

2. LITERATURE REVIEW

Interest rates are associated with the idea that a lender request a premium for undertaking the risk of lending money; hence, the logical argument that interest rate is modeled to be positive (Haksar and Kopp, 2020). Traditionally, a model that gives a negative interest rate due to pricing was considered inefficient. With the current situation, most developed countries' interest rates follow a negative trend; models like the Hull- White and the Bachelier were overlooked due to their acceptance of negative values, while underlying pricing assets tend to be taken into consideration. The Hull- White model in the classic book of Brigo Mercurio, states that there is a possibility of short rates "r" going to 0. However, such assessment exists only in the Gaussian distribution theory, which has a minor impact on pricing derivatives like options. An alternative model to the Hull- White is the Bachelier model, where interest rates can be negative, as proposed by Louis Bachelier (1900).

The following section of the literature review will primarily focus on the theoretical framework of options pricing models. Since this study's main subject is options pricing under negative rates, a historical review of the two concepts (negative rates and options trading) will be first elaborated.

2.1. Historical review of negative interest rates

The great recession caused low growth, low levels of investments, and inflation in a large number of advanced economies. Since then, the central bank has been implementing aggressive monetary policies. Among these, negative interest rates may be the most surprising and, in particular, least comprehended.

The first attempt was from Denmark in 2012. Where most financial advisors were doubtful, the move did not indeed result in any financial system dysfunction. In 2016 the ECB and several central banks in Europe followed the movement. Implementing a negative rate policy usually works with positive rates in regular times. However, there could be some effects on banks where massive lending can be observed and later result in payment defaults.

The idea that short-term rates can be moved below zero was, for a long time in history, inconceivable. After all, why would someone want to be charged for deposing money in an account when they can easily preserve it in cash for zero interest rate? (Altavilla et al, 2019)

Interest rate cuts below zero would logically make everyone with a saving account run to the bank to inquire about cash before registering some loss on deposit. Central banks would therefore witness a cash shortage.

Economists have proposed strategies to implement negative rates and regain banks' firepower. In the middle of the 19th, Silvio Gessel suggested taxation on holding cash (Rosalsky, 2019). In 2009 Greg Mankiw proposed a system based on picking serial numbers on bank notes and declaring them void, making it risky to hold on to cash (Mankiw, 2009).

When rate cuts below zero process without making cash costly to hold was implemented, it changed investors' perspective. Zero was not anymore considered the limit on interest rates. Many investors and consumers were, in fact, willing to pay for the convenience of not preserving cash.

In Switzerland, for example, studies showed that interest rates could go as low as 0.75% without inciting a massive cash demand (Mombelli, 2020). So far, experts have given their view on how far a lower bound might be expended and what it depends on. However, in the end, with all the forecasts, the interest rate world is still unpredictable since many factors impact the implementation of the monetary policy.

2.2. The dynamics behind negative interest rates

Commercial banks must deposit a portion of their liquidity at the Central bank as a mandatory reserve. If there is a cut in rates below zero, it implies that central banks charge commercial banks on any deposit.

Negative rates occur when interest is credited rather than being paid to lenders. This situation can therefore be acknowledged as a money-loss activity. From a certain point of view, investors and commercial banks are punished for holding onto their cash (available in a bank account). To compensate for the loss, banks would strongly prefer to lend to each other under the condition that the interbank lending rate remains less negative than the rate imposed on deposits by the central bank. However, in most cases, central banks still prefer penalizing central banks rather than lending their surplus of money to investors. The reason is that the loss due to the risk on attributed credits remains higher than that caused by central banks.

Negative rates can be conceived as taxes imposed by the central bank on commercial banks in order to incite them to lend money to companies, investors, and households. The main goal when Central banks implemented negative rates was to boost the economy recovering from a recent financial crisis. Negative or low rates increase asset prices by spreading credit money throughout economic agents. Investors will therefore choose riskier instead of safer assets. Moreover, a country's Exchange rate under a negative rate is indirectly depreciated. Clever investors, by exchanging currency, might purchase government bonds of countries where a positive rate policy is still in application. Since there is a depreciation in the Exchange rate, net exports are, on the other hand, boosted.

A pension fund holding a deposit with a commercial bank will seek to purchase assets with a higher return if interest rates drop. The asset's price will increase since it will be highly solicited on the market.

The Central bank aims to boost economic activity and control the spikes in inflation. It can at least be realized in four ways (CBDC, 2021):

- Commercial banks can attribute more loans to households and investors instead of holding on to cash which has now a price to pay
- Households relying on the return on the interest rate could borrow to spend more since interest rates are no longer lucrative.
- The possibility of the fall in demand for the currency might cause its depreciation. As a result, prices of imported goods will increase, which will trigger the demand for the country's cheaper exports.
- Businesses can now invest more since borrowing has become cheaper.

The process of negative interest rates implementation can be summed up into the following steps:

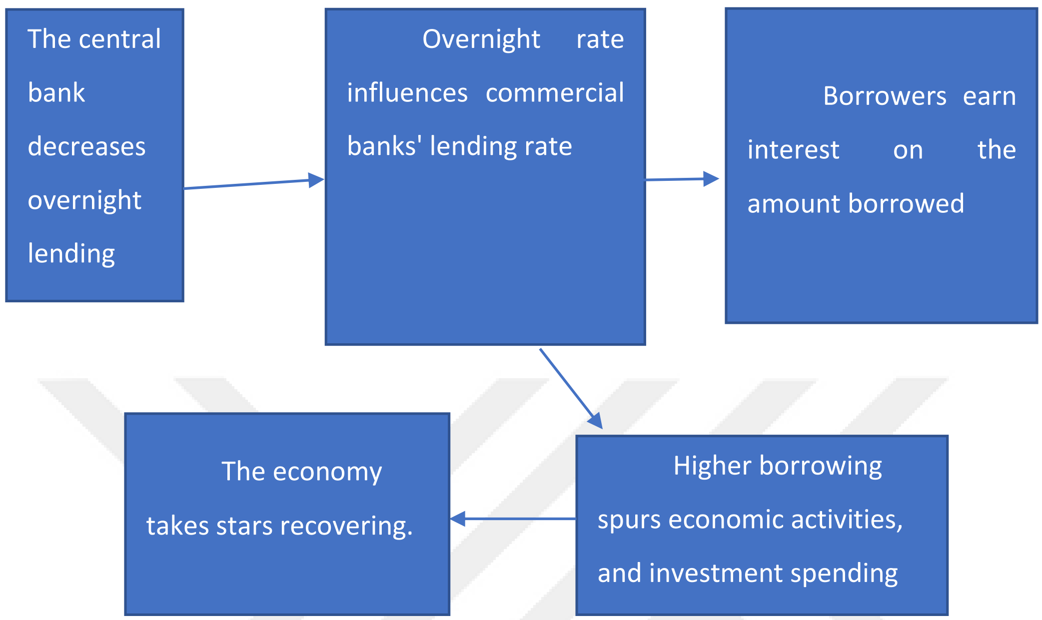


Figure 1:Negative interest rate implementation process ***(Corporate Finance Institute, 2010)***

2.3. Real and nominal interest rates

When talking about the concept of interest rates, it is essential to understand the types of interest rates that are observed in the market. The most popular interest rate is called the nominal interest rate. It calculates the earnings per year on a $100 savings. On the other hand, the real interest rate measures the worth of $100 after one year (if we consider its purchasing power after one year) (Cussen, 2021). The real interest rate that plays an essential role in savings depends not only on the nominal interest rate but also on inflation. The real interest rate becomes negative in a country with a higher inflation rate than a nominal rate. Consequently, the value of households' savings will be low. The opposite case increases savings.

2.4. What are the dangers of negative rates?

The danger one can expect when it comes to negative rate implementation is that there is no certainty at what point financial institutions and households will want to cash in their retained bonds and bank deposit (Inhoffen &Pekanov, 2021). The lower bound is hard to determine, and obviously, reaching this bound could impact the mechanism of the financial system. Other concerns are that low-interest rates could persist instead of low nominal rates. Low actual interest rates can bring the risk of financial instability.

The monetary policy is supposed to operate the following way (CFI, 2010): Stimulate risk-taking and the economy. Monetary policy measures often guide market movement. If securities on the market are volatile due to currency instability, investors can decide to orient their investment in real estate or non-productive financial securities because of their capacity to generate high intrinsic values; a bubble might emerge.

Since the global financial crisis, regulating market prices and spotting excessive financial risk-taking has become the major concern in implementing monetary policy. Central banks react to financial instability in many different ways. Some may use so-called macroprudential measures (direct regulation of risk-taking). Other central banks prefer to raise interest to fight against inflation even though an increase in interest rates before the economy has returned to growth logically delays the return to growth.

Implementing negative rates results in an adjustment of pricing methods. For example, if we consider some underlying options, futures, swaps, caplets, and floorlets, existing classic models like the Black Scholes model do not allow negative rates in their features. Furthermore, in models like the Heston Cox Ingersoll, in practice, the distribution of the mean and variance will be affected when densities approach zero. Many practitioners were obliged to adopt a new pricing approach by adding a shifting parameter on the Black Scholes or by reconsidering a model that was not frequently used on the market like the Bachelier model.

Other inconvenient of rate cuts below zero can be:

-Banks' benefits are reduced: commercial banks' earnings can be affected, which can at the same time affect their profit margin.

-uncontrolled risk-taking: even though the implementation of negative rates might boost the economic activity, an unprecedented flow of funds towards risky assets can be observed in the perspective of obtaining higher yields.

-operational risks: Since negative rates are a new subject, different trading systems are unfamiliar with options pricing under a negative short-term policy. The consequence of this situation might lead to mispricing in derivative products.

The figure below illustrates the 3 months Euribor trend for the last ten years.

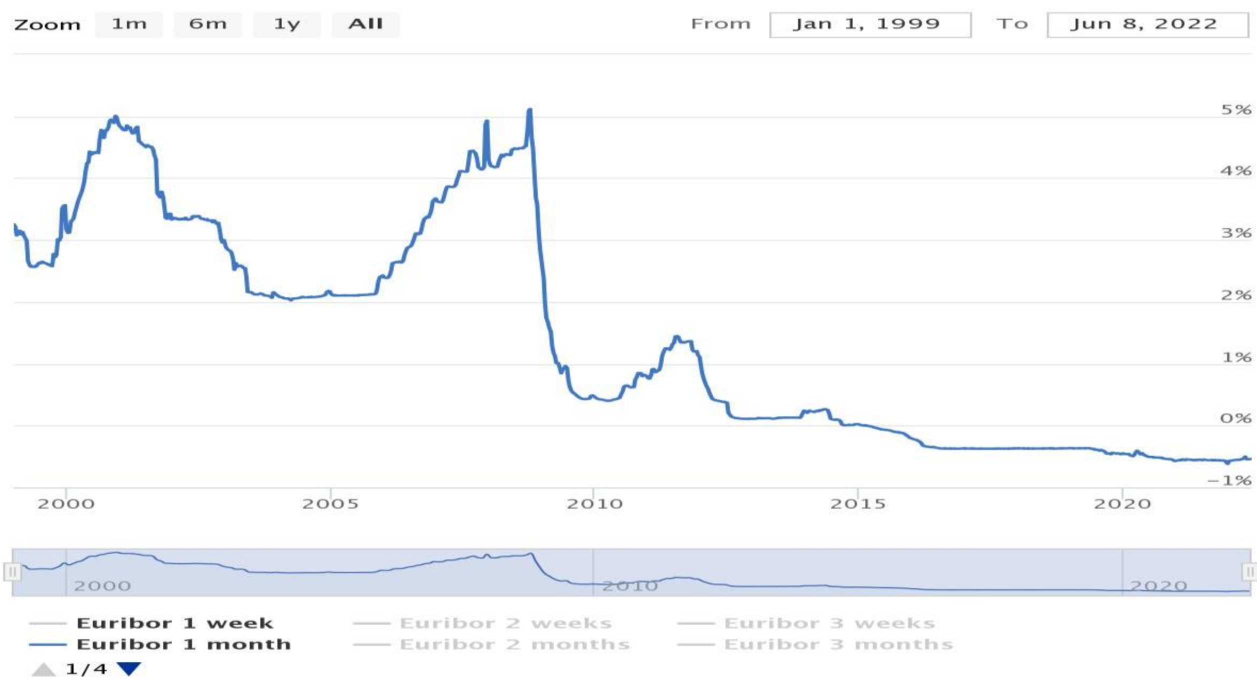


Figure 2:3 months Euribor (https://www.euribor-rates.eu)

2.5. A brief history of options trading

Options trading is often regarded as a new concept compared to another traditional form of investment. After the Chicago Board of Options Exchange (CBOE) was established, options contracts started being known and adopted by several investors. The concept of options trading or contracts goes back to ancient Greece (most probably as long as the mid-fourth century BC).

Options contracts took many forms throughout the time before the CBOE was formed in 1973. Here are some important phases of options trading from ancient Greece until the modern days (Poitrass, 2008).

- Thales and the speculation on olive prices
- The Tulip Mania in the middle of the 17th century.
- Bans on options trading
- Russell Sage and put &call brokers
- The listed options market
- Continued evolution of options trading

2.5.1. Thales and the speculation on olive prices

In the book "Politics," written earlier in the middle of the fourth century BC by Aristotle, Thales of Miletus profited from an olive harvest by speculating that prices would go up at one point in the future. Thales, known for the famous theorem in geometry: "If there is an inscribed triangle in a circle, where one side of the triangle happens to be the diameter of the circle, then the angle opposite to that side is a right angle" had a strong knowledge in astronomy and applied mathematics. By predicting that there would be a vast harvest of olives, he decided to profit from it by elaborating what is called "the first options contract".

However, Thales did not have enough capital to purchase all the olive plantations in his region. As a last resort, he paid a considerable amount of money to the owners of olive plantations to secure the right to exploit them during harvest periods. As predicted by Thales, there was indeed a devastating harvest, and he exercised his rights and made exorbitant profits.

Even though the term was not in practice used at that time, Thales had technically created the first call option in history with olive as the underlying commodity. He exercised his right to use olive presses at a fixed price and was then able to generate profits (Poitrass, 2008).

2.5.2. The Tulip Mania in the middle of the 17th century

In the 17th century, tulips were unbelievably popular. They were regarded as status symbols among the Dutch aristocracy. Their popularity in Europe increased the demand at a dramatic rate.

At this point, calls and puts were starting to get popular in different markets for hedging purposes. For example, put contracts were written on tulips if wholesalers wanted to protect their future profits.

The increased demand for tulips and lack of regulation on the options market caused a bubble. The aftermath caused options to gain a bad reputation throughout the World (Harford, 2020).

2.5.3. Bans on options trading

Despite the bad reputation that options contracts had, they were still used as a hedging instrument by many investors. Throughout history, options have been restricted by several World economic powers. The most notable ban was the one imposed in London, England. Despite establishing a well-structured market for calls and puts in the late 1600s, options trading was banned for more than a century, and the ban persisted until the 19th century.

2.5.4. Russell Sage and put &call brokers

Russell Sage, an American financier, was involved in a significant milestone in the history of options trading. Sage began developing call-and-put options that could be sold over the counter in the United States in the late 1800s. There was no established exchange market then, but Sage's action was a massive advance for options trading.

Sage is also credited with being the first to link the price of an option to the price of the underlying securities and interest rates. Using the put-call parity principle, he constructed synthetic loans by purchasing stock and a related put from a customer. As a result, he was able to lend money to others efficiently. That was a time-consuming process, and the terms of each contract were primarily established by the two parties involved.

The Put and Call Brokers and Dealers Association was created to create networks that would help match buyers and sellers of contracts more effectively. However, there was still no standard for pricing them, and the market severely lacked liquidity. By this time, options trading was definitely on the rise, although investors were still apprehensive due to the absence of regulation (Poitrass, 2008).

2.5.6. The listed options market

With contracts sold over the counter, put and call brokers continued to control the options market virtually. The industry began to standardize, and more individuals became aware of these contracts and their possible applications. At this time, the market was still relatively illiquid, with little activity.

The disparity between what buyers were prepared to pay and what sellers were willing to take was where the brokers made their money, but there was no true, accurate pricing structure, and the brokers could set the spread as wide as they liked.

Despite the Securities and Exchange Commission's (SEC) efforts to bring some oversight to the over-the-counter options market in the United States, by the late 1960s, the market had become unregulated.

The Chicago Board of trade was subject to an unprecedented downfall in trading commodity futures on its exchange, which led the organization to look for new ways of expanding its business. The primary purpose was to develop a diversification strategy and explore more opportunities for members of the exchange to trade.

Several roadblocks were overcome before the Chicago Board of Options Exchange (CBOE) began trading in 1973. Options contracts were adequately standardized for the first time, and a fair market existed in which they could be exchanged. At the same time, the Options Clearing Corporation was formed to provide centralized clearing and ensure contract fulfillment. As a result, many investors' anxieties about contracts not being kept have been alleviated. Trading options became legal over 2,000 years after Thales invented the first call options.

2.5.7. Continued evolution of options trading

At the beginning of its activities, the Chicago board of options trading had only a few available contracts, most of which were composed of call contracts since put contracts were yet to be standardized at that time. Traders were still skeptical about options being able to generate money, hence their low volume of trades at the beginning.

The absence of specific pricing methods of an option mixed with wide spreads implied that the market still lacked liquidity. In 1973, two professors, Fisher Black and Myron Scholes, came up with a pricing formula based on a non-arbitrage condition and made of specified variables that could calculate the price of an option. The formula became known as the Black Scholes pricing formula and is still used nowadays (Poitrass, 2008).

2.6. Options trading and options pricing-related theories

Since options are the main focus of this study, it is relevant to Evoque some terms currently used in the options trading jargon. From the simple concepts of call and put options, the following part will provide the necessary background before further advanced theories related to options pricing.

2.6.1. The components of an option

The concept of option pricing refers to a contract between a buyer and a seller, where the option contract owner bets on future prices of an underlying security or index. On the expiration date, the holder has the right to buy or sell at the strike price but is not obliged to do so (Dheeraj Vaidya, 2009).

The value of options certainly depends on the value of the underlying security. Therefore, the market value of options will increase or decrease based on the underlying security's performance.

Options can be considered as a bet. The terms below are commonly used in options trading (Fidelity Investments, 2015):

- The strike price: the strike price can be considered the price at which the underlying security is bought or sold when exercising the option.
- A call option holder bets on the price rise, whereas a put option buyer bets on the opposite. A call option holder has a bullish view since he bets on an underlying price rise. The buyer in the put option contract has a bearish view and speculates that the underlying security price will drop.
- In–the–money: An in- the- money call option strike price is always under the current stock price. For example, for an investor desiring to purchase a call option at the strike price of 90$, the underlying security is currently being traded at 100$. The investor's position will be in the money by 5$. The call option allows the investor to buy the underlying asset at 90$. The put option works in the same logic as the put option's price is lower than the underlying asset's current price. A put option gives the right to an investor to sell an asset at a certain fixed price in the future. The in-the-money put option will reside in the difference between the actual price and the selling price
- At the money strike: the call and put option are both above the price of underlying security being traded.
- The premium: the price an investor pays to a seller for an option is called the premium. The amount of the premium which is always paid upfront at purchase and is not refundable even when the option is not exercised. The premium is determined by several factors like the amount of the premium concerning the stock price (intrinsic value), the option's time value, and the volatility value.

2.6.2. Options trading strategies

Trading Options implies adopting different strategies according to the type of risk a trader is willing to hedge. The following list describes some popular strategies available on the market (Wallstreet Mojo, 2020).

2.6.2.1. Short put

In this strategy, the price of an underlying is expected to go below the strike price at the expiration date. If the price goes low, the contract holder can purchase the stock. The strategy facilitates a stock purchase at a lower price.

2.6.2.2. Long call

Usually adopted by bullish investors, there is a massive expectation of the underlying to rise in price. Purchasing call is an excellent way of capturing upside potential with very limited down risk.

2.6.2.3. Long put

This strategy is considered a bearish move, where an investor bets on the price to go down. The move constitutes a reasonable risk hedging strategy, especially if there is a risk of over-valued securities in the future.

2.6.2.4. Protective put

As suggested by the name itself, the protective contract is a contract purchased to mitigate the risk on the stock that the buyer already owns. The trader is doubtful about the stock and prefers to hedge the stock holding with a put option to protect himself from any loss. It is similar to owning an insurance policy.

2.6.2.5. Short straddle

Put and call prices are written with the same strike price and expiration date in this example. There is no duty to exercise the put option if the trader makes incorrect forecasts about the stock dropping into the put option and the stock price rises instead. Instead, investors must purchase the stock at a higher price under the call option.

2.6.2.6. Long straddle

The investor buys a long call and a long put for the same underlying asset in this approach. Both have the same expiration date and strike price. When news about an asset spread, traders speculate on anticipated volatility.

2.6.2.7. Credit spread

The credit spread refers to the yield difference and is frequently associated with high-risk trading institutions. It is a term commonly used in the United States, and it refers to the fundamental elements of a US Treasury bond. In this approach, the trader writes a high-premium option contract and then buys a lower-premium option contract on the same underlying. Both alternatives contribute to the credit margin.

2.6.3. Option pricing

The overall option pricing can be broken down into two factors: intrinsic value, which is the value of any option if it is to be exercised based on the spot (current stock price) and the strike price. The other component of option pricing is the time value which also depends on three factors: the time to maturity (how long before the option expires), the volatility (higher volatility increases options' prices whereas low volatility does the opposite), and the interest rate.

Many factors often influence options prices. Before introducing some theories related to pricing, it is important to understand some important factors that have a significant impact on the options price (Bank of America corporation, 2022):

- The quality or the appreciation of the underlying asset
- The return per share and the dividend rate of return of the underlying asset
- Supply and demand for options involving the underlying asset
- Prevailing short-term rates

2.6.3.1. The greeks in options pricing

Predicting what will happen to the price of a single option or a position involving multiple options as the market changes can be complex. The Greeks determine what could happen in pricing changes for moves in a stock, implied volatility. However, it should clearly be understood that the Greeks do not determine the price of an option. There are five major notations: Delta, Gamma, Vega, and RHO for interest rates (The Options Playbook, 2022).

Delta; Delta measures the change in an underlying if the stock moves 1$ in price. Calls are considered to have a positive delta between 0 and 1. That implies that if the stock price moves up, there is no change in pricing parameters, and the call price will go up. Put on the other hand is considered to have a negative delta, between 0 and -1. This implies that if a stock's price goes up and there is a change in pricing dynamics, the option price will certainly go down.

Gamma; Gamma measures the rate at which Delta will change based on a $1 change in the stock price; if Delta is considered as the speed measure at which an option price changes, Gamma can be considered as the acceleration.

Theta; Time decay, or theta, is the option buyer's number one opponent. On the other side, it is usually the best friend of the option seller. For a one-day shift in the time to expiration, theta is the amount that the price of calls and puts will fall (at least in principle).

Vega; Vega measures how to call and put prices will be affected if there is a change in the market's implied volatility parameters. It is regarded as one of the Greek measures that's shaky and over-caffeinated. However, Vega mostly affects the "time value" of an option price. Technically, if the implied volatility increases, the option's intrinsic value also increases.

RHO; Last but not least, the greek "RHO" measures the amount an option value will change if there is a change in a one percentage point base of interest rates. This parameter, even though it is not talked about a lot, it has a significant impact on an option payoff, especially now that most European countries have negative rates now and classic pricing models have to be adjusted

2.6.4. Empirical studies done on derivative pricing under negative interest rates

In February 2016, Deloitte, prior to the surge of negative interest rates, proposed the shifted Black Scholes, which consists of adding a shift to the traditional Black Scholes to allow negative input of negative interest rates. However, the model does not tackle the volatility observed in the market; hence the use of a shifted SABR model that is almost similar to the shifted Black Scholes but includes necessary features for the implied volatility. Hagan and collaborators introduced the SABR model in 2002. The model owes its popularity to the fact that it incorporates the implied volatility features of the Black Scholes. The derivation of the SABR model implies certain truncations, which leads to some minor errors. An equivalent way to see the breakdown of the SABR model is to price butterfly spreads which, due to the positivity of the convexity of the cap payoff, should remain positive. Despite its considerable errors, the SABR is still preferably used when pricing the vanilla types of options.

The classic Black Scholes model assumes a constant normal volatility parameter. Practitioners agreed to an implied volatility model to tackle the volatility observed in the market. Local volatility dynamics can be extended to calibrate or measure the implied volatility in Black's formula, but they will still lead to erroneous results in risk metrics dynamics. The SABR model proposed by Hagan in 2002 is a two-factor model that follows the Brownian Motion and incorporates some approximative dynamics of the implied volatility observed in the market. The model is unreliable in a negative rate environment unless it has added a shifting parameter. Luuk

Hendrick Frankena (2016) analyzed three different solution methods to cope with negative rates while hedging options. The normal Bachelier has a normal distribution and presents a considerable advantage because it does not have to add a shifting parameter to cope with negative rates. The normal SABR model can be used in a negative rate context; the only disadvantage was the assumption of positive probability on significant negative rates. As a solution, Frankena proposed three boundary models for their capacities of modeling rates from the entire real line without introducing some additional parameters. Pricing and hedging European- type options with different expiries require accurate calibration. The SABR can only solve this problem by extending the model with dependent parameters.

Agustin Pineda (2017) analyzed derivative pricing formulas. Among them, arbitrage-free models are suitable for market pricing curves. The shifted SABR model was designated as well-performing among alternative competitors. The models showed outstanding results for both in-sample and strike out-of-sample analysis. However, when dealing with out-of-sampling maturity, the performance seemed to alter.

A study of six models on options traded on the SP500 was done by Bruce Tsoe Jin (2019): the standard Black Scholes model, the black S- Vasicek model, the BS cox Ingersoll model, the standard Heston Hull White model, the Heston- Vasicek model, and the Heston- Cox İngersoll model. The main purpose was to highlight how models perform under low or negative rates and how accurate they could be when it comes to pricing options. Among the analyzed models, those that assume a cox Ingersoll interest rates types do not perform well in negative rates. The simulated prices present more errors than other models without CIR interest rate calibrating models. The standard Black Scholes model had the worst performance because it does not incorporate the volatility present in the market. Plus, it is known to accept positive values unless a shifting parameter is added. The standard Heston model and the Heston Vasicek had the best performance amongst all the studied models. Bruce Tsoe Jin (2019) suggested adding a shift on the CIR model and a Heston stochastic volatility model to give accurate derivatives prices in periods of low-interest rates.

2.6.5. Theoretical framework related to options pricing

Before analyzing how the pricing process is done under a negative rate environment, it is important to comprehend some concepts related to asset pricing. It is even more relevant to review some existing models and why practitioners for the negative rate environment do not prefer them.

2.6.5.1. Asset's dynamics and important theories

Dynamics behind asset pricing require them to follow the Brownian Motion movement. Prices for assets are often found after transformations of SDEs into PDEs. The following section will describe the dynamics behind options pricing and some important theories related (Grzelak&Oosterlee, 2019).

The Brownian motion;

In order to understand the logic behind the Brownian, an example of a coin can be given:

Let us assume that every time a coin is flipped in the air and lends on heads, a person "x" earns a dollar. When the same coin lends on tale, the same person "x" will lose a dollar. Mathematically it can be represented by: Ri:$\pm\mathbf{1}$ (randomness or changes).

Outcomes will therefore be summed up by Si: $\sum_{J=1}^{i} \boldsymbol{Rj}$ (3.0)

The Brownian Motion also includes two popular properties in probabilities: the Markov property and the martingale property. Both insist on the randomness of a variable. The Markov property insists that past solutions do not impact or influence future solutions. The martingale property states that an expected outcome always equals the present outcome. With the martingale property, our example's expected return or outcome can mathematically be represented by: $\boldsymbol{E}\left[\frac{Sj}{Si}, \boldsymbol{i} < \boldsymbol{j}\right] = \boldsymbol{Si}$. (3.1)

To make the randomness of every solution continuous, we use the Brownian motion. To demonstrate to continuity of the randomness, we divide it into "n" over the time "t."

The Brownian Motion has the following properties:

-E[X(t)]=0 and E=[X(t²)]=t

-The Brownian Motion is continuous

-The Brownian Motion obeys the martingale property: E[X(t)/X(τ), τ<t]= X(τ)

- The Brownian Motion follows a normality relation where τ< t with a mean reversion "0" and a standard deviation $\sqrt{t} * \tau$.

One approach to developing a more realistic model for asset price, which still retains the properties of the random walk as the market efficiency suggests it, is to derive a continuous-time model from random walks. One way to do it is to take the limit as the number of jumps in any unit of time goes to infinity. If this procedure is done, there is necessary to scale down or shrink the size of jumps. For instance, the variance will go to infinity.

Let σ^2= var (Xj)

Lim var (Sm) = $\lim_{m\to\infty} var\ (\sum_{j=1}^{m} Xj)$ (3.2)

By independence we will have = $\lim_{m\to\infty} \sum_{j=1}^{m} var\ (Xj)$

$$= \lim_{m\to\infty} m\sigma^2 = \infty \quad (3.3)$$

The jumps Xj of size 1 must be scaled down for the limit to be sensible. The jumps should reduce by dividing by some factor.

$$\frac{Xj}{\alpha}$$

The question that can arise here is how α should be chosen.

The central limit theorem suggests that:

$\alpha = \sqrt{m}$: $\lim_{m\to\infty} (\sum_{j=1}^{m} \frac{Xj}{\sqrt{m}})$ = $\lim_{m\to\infty} \sum_{j=1}^{m} var\ (\frac{Xj}{\sqrt{m}})$ (3.4)

The probability theory allows us to use: var $(\alpha x) = \alpha^2 var(x)$ (3.5)

By independence, we will then have

$\lim_{m\to\infty} \sum_{j=1}^{m} \frac{1}{m} var\ (Xj)$ = $\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} \sigma^2$ = $\lim_{m\to\infty} \frac{1}{m} m\sigma^2$ = σ^2 with $\sigma^2 < \infty$ (3.6)

Suggests that $\sqrt{m}$ is the right scaling. Here the purpose is to consider the limit as the number of time steps m per unit gets large. For any real-time t≥ $\mathbf{0}$ the number of jumps will be some integer closer to mt. Scaled random walk is going to be defined by $s_t^{(m)} = \sum_{j=1}^{[mt]} \frac{Xj}{\sqrt{m}}$

Here $[\boldsymbol{m}]$ is the floor function, which is the greatest integer less than X, so for example, $[\mathbf{3,7}] = \mathbf{3};\ [\mathbf{0,9}] = \mathbf{0}$.

The Brownian Motion is a continuous-time stochastic process conventionally denoted W(t) with the following properties:

0. W(0)= 0 with probability 1.

1. The sample paths are continuous with probability 1

2. For t$< \boldsymbol{s} < \boldsymbol{r}$ W(s)- W(t) is independent of W(r)- W(s)

3. For any t,s $\geq \mathbf{0}$. W(t)- W(s) is normally distributed with a mean 0

The independence of jumps in the random walk passes to the independence of increments in the limit. The random walk converges to Brownian Motion in an exact sense $\boldsymbol{s_t^{(m)}} \rightarrow \boldsymbol{W(t) as\ m} \rightarrow \infty$. In the sense of weak convergence of stochastic processes.

The Brownian Motion with a drift

A model should support a trend to be a suitable representation of asset prices. The Brownian Motion is not sufficient because the expectation stays 0 all the time.

In order to remedy this, a term that represents a trend has to be added to the Brownian motion. It is also desirable to add a volatility factor $\boldsymbol{\sigma}$. Brownian Motion with drift is the stochastic process W(t,$\boldsymbol{\mu}, \boldsymbol{\sigma}$)= $\boldsymbol{\mu t} + \boldsymbol{\sigma W(t)}$ (3.7)

$\boldsymbol{\mu t}$ is the drift term, and time allows adjuments of the volatility of the Brownian Motion term (Etienne and Vallois,2007).

The log-normal model

In order to remedy some outstanding of the Brownian motion, there should be some modifications. Like the random walk model, the Brownian Motion has a significant probability of attaining negative values. Adding a drift term reduces the probability but does not eliminate it. More serious is the limitation shared by both the random walk and Brownian Motion that the size of the jumps does not scale with the price level. Here the idea is to expand the model to remedy defects while still preserving the elements of the market efficiency in the independence of increments.

Mathematically we can try to interpret it this way:

Let's denote S(t) the asset price. We consider changes log (S(t))

Let $\boldsymbol{\varphi(t) = \log(S(t))}$ (1)

by the law of logarithms will have S(t)= $\boldsymbol{e^{\varphi(t)}(2)}$.

If $\boldsymbol{\varphi(t+1) = \varphi(t) + \delta}$(the additive jump or shocks then modelling shocks or jumps applied in the log S(t)

S(t+1)= $\boldsymbol{e^{\varphi(t+1)}}$

=$\boldsymbol{e^{\varphi(t)+\delta}}$

=$\boldsymbol{e^{\delta}.\, e^{\varphi(t)}}$

= $\boldsymbol{e^{\delta}}$.S(t) (3.8)

Here $\boldsymbol{e^{\delta}}$ is the price multiplicative price jump that scale with the price itself so the price jump can be written: S(t+1)- S(t)= $\boldsymbol{e^{\delta}S(t) - S(t)}$ (3.9)

= $\boldsymbol{(e^{\delta} - 1)S(t)}$ this relationship holds no matter what the overall price level. The price could be multiplied by any factor and still the price jump would be the constant $\boldsymbol{e^{\delta} - 1}$ times the initial price. A fixed additive jump in log(S(t)) implies a price jump that scale with the price level.

A model in which the dynamics involve jumps of fixed size (on average) in log(S(t)) would have the sensitivity to the overall price level that is expected. This suggests the idea of modelling log(S(t)) as Brownian Motion rather than S(t) itself. This argument can also be verified by stationarity concept. By definition a strict stationarity means the multivariate distribution of returns does not change with time. Empirically, asset returns tend to be stationary over reasonable periods of time. As a matter of fact in financial economics and econometrics, the consesus view is that asset return are stationary, at least as a first approximation.

Therefore it’s reasonable to choose assets price models that lead to stationary results. These considerations lead to us a consider modelling log(S(t)) as a Brownian Motion rather than S(t) itself:

Log S(t)= W(t) for returns we will have

Log ($\boldsymbol{\frac{S(t)}{S(t-1)} = \log(S(t)) - \log(s(t-1) = W(t) - W(t-1)}$ (3.10)

From the properties of Brownian Motion the sequence of brownian differences W(t)-W(t-1), W(t+1)-W(t),…… are stationary as well as uncorrelated (independent in fact)

Thus modelling log(S(t)) as Brownian Motion accomplishes 3 things:

1) Average price jumps are proportional to the price level
2) Returns are stationary
3) Returns are uncorrelated

It's standard to add a drift term and a volatility factor $\boldsymbol{\sigma}$ to this model.

Log(S(t)) = W(t)+ $\boldsymbol{\sigma W(t)}$ (3.11)

Also, an initial value S(0) should be included: log(S(t))= log(S(0))+$\boldsymbol{\mu t + \sigma W(t)}$

We may then exponantiate both sides of this equation: S(t)= S (0). $\boldsymbol{e^{\mu t+\sigma W(t)}}$. This can be recognized as a model for asset prices or the Geometric Brownian Motion.

The simulation of the Brownian Motion gives the following results for a stock path after its simulation.

If we consider for example a stock value of $\boldsymbol{S_0 = 100}$, μ($\boldsymbol{the\ drift\ term}$) = $\boldsymbol{0.1\ and \sigma = 0.3}$ after calibration in python

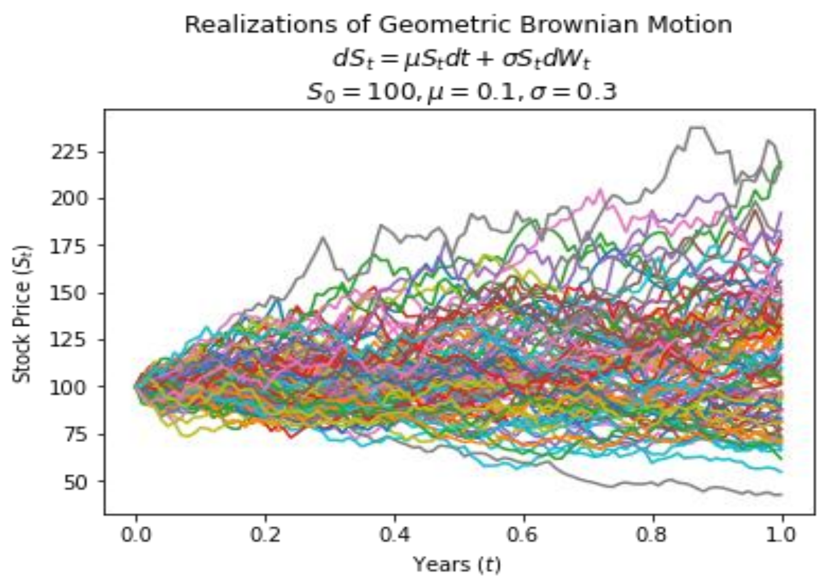


Figure 3:Asset's paths simulation under the Brownian Motion

2.6.5.2. Martingale approach for option pricing

A stock process X={X(t);t≥ $\mathbf{0}$} is a martingale Wrt if :

-X is adapted to the filtration $\boldsymbol{F(t)} \geq \boldsymbol{o}$,

-for all t, $\mathbb{E}(|\boldsymbol{X(t)}|) < \infty$

-for all S and t with S> $\boldsymbol{t}$ we have $\mathbb{E}$(X(t) $\boldsymbol{F(S)} = \boldsymbol{X(S)}$).

Finding a risk free drift.

Let's assume an asset process given by $\frac{\boldsymbol{dS(t)}}{\boldsymbol{S(t)}} = \mu \boldsymbol{dt} + \boldsymbol{\sigma dW^{p}(t)}$ (3.12) and a bank account: $\frac{\boldsymbol{dM(t)}}{\boldsymbol{Mt}}$=rdt. The objective is to find a discounted payoff under the risk neutral measure Q so the process $\frac{\boldsymbol{S(t)}}{\boldsymbol{M(t)}}$ is martingale. In practice we will have: $\mathbb{E}^{\boldsymbol{Q}}(\frac{\boldsymbol{S(t)}}{\boldsymbol{M(t)}}|\boldsymbol{F(t)} = \frac{\boldsymbol{S(t)}}{\boldsymbol{M(t)}}$. The Dynamics of F=$\frac{\boldsymbol{S(t)}}{\boldsymbol{M(t)}}$ are found from Ito's lemma:

$$D\frac{S}{M} = fSdS + fMdM + \frac{1}{2}fSS(dS)^2 + \frac{1}{2}fMM(dM)^2 + fSMdMdS$$

$=\frac{1}{M}dS - \frac{S}{M^2}dM + 0 + 0 + 0$ with replacement the equation

$$=\frac{1}{M}(\mu Sdt + \sigma SdW) - \frac{S}{M^2}rMdt$$

$=\frac{S}{M}(\mu - r)dt + \frac{1}{M}\sigma SdW^{p}$ (3.13). μ Needs to be equal to $\boldsymbol{r}$ so that all the Dynamics of $\frac{\boldsymbol{S(t)}}{\boldsymbol{M(t)}}$ can become driftless.

The motion of Martingale provides an probable method for deriving equations related to derivative pricing (the starting point of every pricing equation):

V $(\boldsymbol{S(t)}, \boldsymbol{t}) = \boldsymbol{M(t)}\mathbb{E}\left[\frac{\boldsymbol{1}}{\boldsymbol{M(t)}}\boldsymbol{V(S(T), T)}|\boldsymbol{F(t)}\right]$.(3.14) With $\boldsymbol{M(t)}$ being the money saving account at the time $\boldsymbol{M(t)}$=1 and $\boldsymbol{V(S(T), T)}$ considered as a payoff.

$$\mathbb{E}^{Q}\left[\frac{1}{M(T)}V(S(T), T)|F(t)\right] = \frac{V(S(t),t)}{M(t)}. \quad (3.15)$$

The payoff discounted to the money savings account is a martingale and Ito's lemma can be used to find its dynamics.

$$\text{D}\left(\frac{\boldsymbol{V(S(t),t}}{\boldsymbol{M(t)}} = \frac{\boldsymbol{1}}{\boldsymbol{M(t)}}\boldsymbol{dV} - \boldsymbol{r}\frac{\boldsymbol{V(S(t),t)}}{\boldsymbol{M(t)}}\boldsymbol{dt}.\right.$$

For an infinitesimal change, dV of $\boldsymbol{V(S(T),T)}$ we find:

$$\text{DV}=\left(\frac{\boldsymbol{\partial V}}{\boldsymbol{\partial T}} + \boldsymbol{rSt}\frac{\boldsymbol{\partial V}}{\boldsymbol{\partial S(t)}} + \frac{1}{2}\boldsymbol{\sigma^2 S}\left(\boldsymbol{t}\right)^2\frac{\boldsymbol{\partial^2 V}}{\boldsymbol{\partial S(t)^2}}\right)\boldsymbol{dt} + \boldsymbol{\sigma S(t)dW(t)^Q}.$$

As $\frac{\boldsymbol{V(S(t),t)}}{\boldsymbol{M(t)}}$ should be a martingale. The Martingale theorem suggests that the payoff discounted to the money savings account can not contain any dt terms. Which implies that (S=S(t), M=(M(t)).

$$\frac{\boldsymbol{1}}{\boldsymbol{M}} = \left(\frac{\boldsymbol{\partial V}}{\boldsymbol{\partial T}} + \boldsymbol{rS}\frac{\boldsymbol{\partial V}}{\boldsymbol{\partial S}} + \frac{1}{2}\boldsymbol{\sigma^2 S^2}\frac{\boldsymbol{\partial^{2}V}}{\boldsymbol{\partial S^2}}\right)\text{-r}\frac{\boldsymbol{V}}{\boldsymbol{M}}\text{=0} \quad (3.16)$$

By multiplying bothside, gives us the Black Scholes pricing PDE.

Stock Paths under the martingale approach can be represented under respectively "P" and "Q" measures. Like it was question for the geometric brownian motion, stock paths are simulated by taking a specific number of paths. Simulations can be done on more than 10 thousands paths depending on the timeline and the accuracy. For this case, only a thousand number of paths was considered.

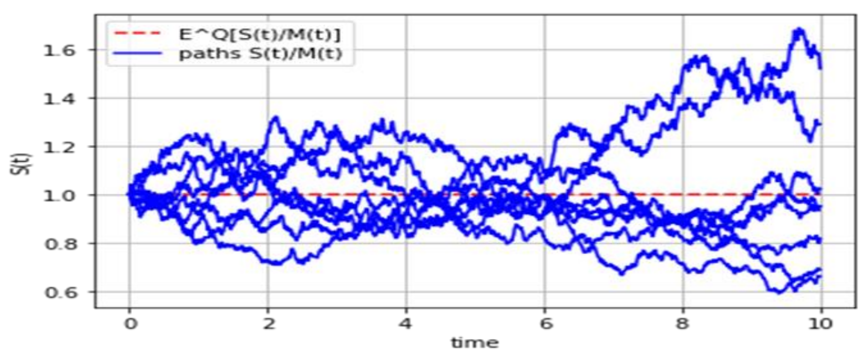


Figure 4: Asset's paths under the Martingale measure "Q"

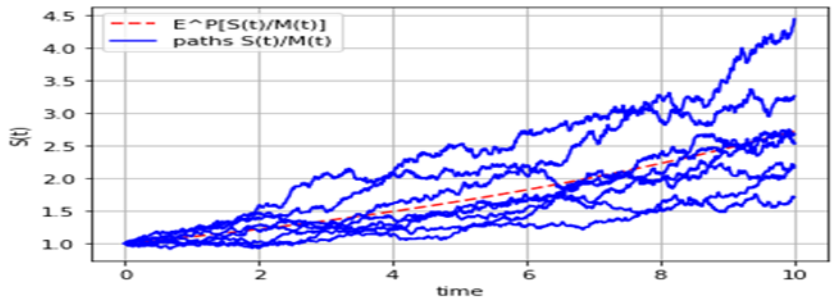


Figure 5: Asset's paths under Martingale measure "P"

2.6.5.3.Ito's lemma

Let's assume a differential equation:

$$\boldsymbol{dx = a(x,t)dt + b(x,t)dz} \quad (3.17)$$

Where dz is a weinar process or follows a geometric brownian motion, a and b are function of x(variable) and t(time). An ito process is a generalized weinar process in which the parameters a and b are function of the value of the underlying variable x and time t: both the expected drift and the volatility can change over time. A derivation of an ito's lemma consist into some sort of series called "taylor series"

According to ito's process, a function G of x and t obeys the following process:

$$DG=\left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1\partial^2 G}{2\partial x^2}b^2\right)+\frac{\partial G}{\partial x}bdz \quad (3.18)$$

Combined with the most used model for stock behavior:

DS= μSdt+δSdz, (3.19) ito's process can be interpreted the following way:

$$DG= \left(\frac{\partial G}{\partial S}\mu s + \frac{\partial G}{\partial t} + \frac{1\partial^2 G}{2\partial S^2}\delta^2 S^2\right)+\frac{\partial G}{\partial S}\delta \text{ sdz}. \quad (3.20)$$

The ıto's derivation is often used while deriving options pricing models such as the famous Black Scholes model, the Heston model and others (Grzelak, 2009)

2.6.5.4. Feynman –Kac formula

Feynman introduced a linkage between partial differential equations and stochastic processes. It provides a method for solving certain PDEs by simulating random paths of stochastic process. Lets's assume the following PDE: $s\frac{\partial V}{\partial t} + \hat{\mu}(X,t)\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 V}{\partial X^2} = 0$ (3.21).

Subject to the final condition V (x, t) =η(x) then the Feynman formula becomes: V (x, t) = 𝔼(ηX(T)|F(t)). The first part of the equation is regarded as a payoff and the second part is an expectation of a payoff.

***Proof of the Feynman kac formula**

Previously the PDE for V (X, t) was given. Therefore Ito's dynamics for V are determined by:

$$d\frac{\partial V}{\partial X}\boldsymbol{dx}+\frac{\partial V}{\partial X}\boldsymbol{dx}+\frac{\partial V}{\partial T}\boldsymbol{dt}+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\boldsymbol{dx})^2 \quad (3.22)$$

$$dV=(\frac{\partial V}{\partial x}\boldsymbol{dx}+\hat{\mu}(\boldsymbol{x},\boldsymbol{t})\frac{\partial V}{\partial x}+\frac{1}{2}\widehat{\boldsymbol{\sigma}^2}\left(\boldsymbol{x},\boldsymbol{t},\frac{\partial^2 V}{\partial x^2}\right)\boldsymbol{dt}+\hat{\boldsymbol{\sigma}}(\boldsymbol{x},\boldsymbol{t})\frac{\partial V}{\partial x}\boldsymbol{dW}(\boldsymbol{t}) \quad (3.23)$$

(by Ito's table) If this equation is observed attentively the dV equation's part has a component of the PDE given before for the value of V(X,T). Our new equation for dV will therefore become:

$$dV=\hat{\boldsymbol{\sigma}}(\boldsymbol{x},\boldsymbol{t})\frac{\partial V}{\partial x}\boldsymbol{dW}(\boldsymbol{t}) \quad (3.24)$$

By integrating both sides the equation will become:

$$\int_t^T \boldsymbol{V}(\boldsymbol{X}(\boldsymbol{T}),\boldsymbol{T})-\boldsymbol{V}(\boldsymbol{X},\boldsymbol{T})=\int_t^T \hat{\boldsymbol{\sigma}}\,(\boldsymbol{x},\boldsymbol{t})\frac{\partial V}{\partial x}\boldsymbol{dW}(\boldsymbol{t}) \quad (3.25)$$

Technically the term $\frac{\partial V}{\partial x}\boldsymbol{dW}(\boldsymbol{t})$ is the summation of $\sum\boldsymbol{\sigma}\frac{\partial V}{\partial x}\boldsymbol{W}(\boldsymbol{ti}+\mathbf{1})-\boldsymbol{W}(\boldsymbol{ti})$ if we solve the equation by adding the expectation sign we observe that the two brownian motions are equal to zero according to the rule of independent increments of the brownian motion. After adding the expectation sign to our initial integration, the value of V(x,t) will be:𝔼 (V(X,T),T)=𝔼(η(X(T))) (Grzelak&Oosterlee,2019)

2.6.5.5.Filtration concept

Like it was observed earlier in this work, a stochastic process is a set of random variable indexed by a time variable "t". If we consider a time line on a calender, days: T1, T2, Tn up to today, an asset's value follows what we call stochastic process X(t). The past is known hence we can easily analyse historical asset paths.

For the future, the prices can not be precisely estimated, but a simulation can be done thanks to some asset distribution. By considering assets with same distribution, an attempt to future prices simulation can be made especially if one of the assets has known historical datas.

The filtration process can be noted: $\boldsymbol{\mathcal{F}(t_0)} \subseteq \boldsymbol{\mathcal{F}(t_1)} \subseteq \boldsymbol{\mathcal{F}(t_2)} \dots . \subseteq \boldsymbol{\mathcal{F}(t_n)}$. When a stochastic process X(t) is $\boldsymbol{F(t_0)}$ measurable this implies that at a time point $\boldsymbol{t_0}$ (starting point) the value or the process of an asset is known. X(T1) has $\boldsymbol{F(t_1)}$ as a filtration factor and is measurable but has future realisations which are not yet specific at time $\boldsymbol{t_0}$ mathematically we can express it the following way:

$\mathbb{E}\big(\boldsymbol{S(t)}\big|\boldsymbol{\mathcal{F}(t_0)}\big) \neq \boldsymbol{S(T)}$ ***but*** $\mathbb{E}(\boldsymbol{S(t)}|\boldsymbol{\mathcal{F}(T)} = \boldsymbol{S(T)}$ The second equation is measurable but still stochastic.

Filtration and measure changes are very important in assets pricing but a wrong usage leads to errors in prices determination. The expression $\boldsymbol{\mathcal{F}(T)}$ is said to be measurable if at time $\boldsymbol{t \geq T}$ its realisation is known.

2.6.5.6.Option pricing using the conditional expectation

Let us use the tower property of the expectation with an example of a stochastic differential equation for a stock price: $\boldsymbol{dS(t) = rS(t)dt} + \mathrm{J}\, \boldsymbol{S(t)dW^Q(t)}$ (3.26).

Where J represents a certain stochastic volatility which has a log normal distribution. After standard calculations of Ito’s lemma we obtain the following equation for S(T): $\boldsymbol{S_0 \exp((r - \frac{1}{2}}\mathrm{J}^2)\boldsymbol{T} + \mathrm{J}\boldsymbol{W^Q(T)})$ (3.27).

Since the stock S(t) contains the stochastic term of volatility J it’s non trivial to determine the closed- formula fort he value of a vanilla option. The best solution is to apply the tower property of iterated expectation to find a solution for the European option. By the tower property, using $\mathbb{E} = \mathbb{E}^{\boldsymbol{Q}}$, the call value of an option can be written as a discounted expectation with (Grzelak&Oosterlee,2019):

$\mathbb{E}[\boldsymbol{\max S(T) - K, 0)}|\ \boldsymbol{\mathcal{F}(t_0)}] = \mathbb{E}[\mathbb{E}\,\boldsymbol{\max(S(T) - K, 0)}\,\mathrm{J}|\boldsymbol{\mathcal{F}(t_0)}]$ (3.28).

The variance process of the inner expectation is equivalent to the Black Scholes solution conditioned on the realisation of the variance process. The calculation of the inner expectation is equal to the Black Scholes solution with a time dependent volatility. In practice for a given realization of Y(t), $t_0 \leq t \leq T$ the value of an underlying S(T) is given by the relation:

$$S(t_0)\exp(\left(r - \frac{1}{2}J^2\right)(T - t_0) + J\left(W^Q(T) - W^Q(t_0)\right)) \text{ (3.29).}$$

If we have to solve the inner expectatiopn here we have to proceed by:

$$\mathbb{E}[\max S(T) - k, 0)| \ J = j] = S(\ t_0)e^{-r(T-t_0)} \ fN(0,1)d_1 - KfN(0,1)(d_2)$$

With the variables d1 and d2 having the following values:

$d_1 = \frac{log\frac{S(t_0)}{K} + (r + \frac{1}{2}J^2)(T - t_0)}{J\sqrt{T - t_0}}$, $d_2 = d_{1\_}J\sqrt{T - t_0}$ here F N (0,1) is the standard normal cumulative distribution function. These results can be substituted into the main equation which can give (Grzelak&Oosterlee,2019):

$$\mathbb{E}\left[\max(S(T) - K, 0)\right] = \mathbb{E}[S(t_0)e^{r(T-t_0)} fN(0, 1d_1) - KfN(0,1)(d_2)\right]$$

$$= S(t_0)e^{r(T-t_0)}\mathbb{E}[fN(0,1)(d_1)] - K| \ \mathbb{E}[fN(0,1)(d2)]. \text{ (3.30)}$$

Option pricing under these dynamics require a transformation into a cumulative distribution function. The most challenging part is the expectation part due to some factors being function of the volatility. The best way to deal with the expectation is to use the montecarlo simulation.

2.6.5.7.The concept of the numeriare

When dealing with some sophisticated SDEs for options pricing it's sometimes necessary to impose an appropriate measure transformation in order to reduce complexity. In financial mathematics a numeriare is a tradable asset or entitty that can serve as a reference for prices of other tradable assets. (Grzelak&Oosterlee, 2019)

Under the appropriate numeriare a stochasic process is a martingale since working with a martingale is favorable and help a process being free of drift (note that a processes free of drift can still have an implied volatility strcture it's however considered to be practical to work with.)

Let us consider a tradable asset with X(t) as process the corresponding martingale property are as follow: the risk neutral measure is associated with the Money saving account, M(t) as the numeriare, $dX(t) = \mu^{-Q}(t)dt + \overline{\sigma}(t)dW^Q(t)$ (3.31) if we add the numeraire we get:

$\mathbb{E}^{Q}\left[\frac{X(t)}{M(t)}\middle|\, f(t_0)\right] = \frac{X(t_0)}{M(t_0)}$ (3.32). The Girsanov theorem (changing from one measure to another). (Grzelak&Oosterlee,2019)

Let's assume the following process for a stochastic process X(t) defined by: $dX(t) = \mu^{A}\big(X(t)\big)dt + \sigma(X(t)dW^{A}(t)X(0) = X_0$.

The Brownian Motion dW^{A} is under the measure Q^{A}, $W^{A}(X(t) and\ \sigma(X(t)$ follow the Lipschitz continuity condition for a drift $\mu^{B}\big(X(t)\big)$ for which the ratio Y(t)=$\frac{\mu^{B}\,(X(t))-\mu^{A}(X(t))}{\sigma(X(t))}$ (3.13), we can define the measure Q^{B} by:

$\frac{dQ^{B}}{dQ^{A}}|\ f(t) = \exp(-\frac{1}{2}\int_0^t Y^2\,(S)dS + \int_0^t Y(S)dW^{A}(t)(3.33)$.

After calculations, the measure Q^{B} is equivalent to the measure Q^{A}. The stochastic process $W^{B}(t)\ defined\ by$: $dW^{B}(t) = -Y(t)dt + dW^{A}(t)$ is a Brownian Motion under the measure Q^{B} and finally the process X(t) under Q^{B} can be written as:

$dX(t) = \mu^{B}(X(t)dt + \sigma\big(X(t)\big)dW^{B}(t), X(0) = X_0$. (3.34)

To emphasize more on the change of numeraire, let's take an example and look in real life how the pricing formula of the Black Scholes model behaves under different measures (Grzelak&Oosterlee, 2019):

The Black Scholes model with the Money saving account and a standard log normal process is defined by: $ds(t) = rS(t)dt + \sigma S(t)dW^{Q}(t)$; $dM(t) = rM(t)dt$. We want to determine the price of the following derivative defined by the equation: $V(t_0) = \mathbb{E}^{Q}[\frac{1}{M(T)}\max(S^2(T) - S(T)K, 0|\ f(t_0)]$. In this case the first step is to choose an asset to assimilite our portfolio. In practice there's a change of numeraire from the Money-savings account to the stock by defining the following Radom-Nikodym derivative: $\frac{dQ^{S}}{dQ} = \frac{S(T)}{S(t_0)}\frac{M(t_0)}{M(T)}$ which is equivalent to $dQ^{S} = \frac{S(T)}{S(t_0)}\frac{M(t_0)}{M(T)}$. Under the expectation we get:

$V(t_0) = \mathbb{E}^{Q}\left[\frac{1}{M(T)}\max(S^2\,(T) - ST\Big)K, 0)|\ f(t_0)\right.$ (3.35)

$= \int_{\tau}^{0}\frac{1}{M}\max(S^2 - SK, 0)dQ$ (3.36)

$= \int_{\tau}^{0} \frac{1}{M} (S^2 - SK, 0) \frac{S(t_0)M(T)}{S(T)M(t_0)} dQ^S$ (3.37)

$= \mathbb{E}^S \left[\frac{1}{M(T)} \max(S^2(T) - S(T)K, 0\right) \frac{S(t_0)M(T)}{S(T)M(t_0)} | f(t_0)$ The final equation under the numeraire stock becomes: $\mathbb{E}^S[\max(S(T) - k, 0) \frac{S(t_0)}{M(t_0)} | f(t_0)]$. (3.38)

2.6.6. Options pricing models.

Models for pricing options exist in a huge number. Practitioners have for a long time been discovering models each with different features than the other. In this section, some pricing models are going to be reviewed. As they both aim to price with accuracy, each one of them has its own speciality.

2.6.6.1.The Black Scholes model

The ito's lemma process can help to derive the Black Scholes model and obtain it's PDE equation:

Let's consider S(t) an asset's price and V(s,t) an option price.With the stochastic formula of a stock's price behavior(3.3):

dS=μSdt+δSdz, under the following assumptions:

$\pi = a\ position\ taken\ for\ a\ given\ underlying\ asset$

$$\pi = V - \Delta S \text{ (The hedging factor) (4.0)}$$

(4.0) can also be expressed: $d\pi = dV - \Delta ds$. With the multidimensional derivation of Ito's lemma we will have (Chalamandaris& G. Malliaris,2009):

DV= $\frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1\delta^2 S^2 \partial^2 V}{2\partial S}$ (4.1)

D$\pi = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1\delta^2 S^2 \partial^2 V}{2\partial S} - \Delta dS$ (4.2)

dπ= $\frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1\delta^2 S^2 \partial^2 V}{\partial S^2} dt - \Delta ds$ (4.3)

d$\pi = \frac{\partial V}{\partial S} - \Delta) dS + (\frac{\partial V}{\partial t} + \frac{1\delta^2 s^2 \partial V}{\partial V^2}) dt$ if we choose $\Delta = \frac{\partial V}{\partial S}$ then

d$\pi = (\frac{\partial V}{\partial t} + \frac{1\delta^2 S^2 \partial^2 V}{2\partial S^2}) dt$. To avoid arbitrage, the equation is often interpreted this way

$d\boldsymbol{\pi} = \boldsymbol{r\pi dt}$ ***if we replace π by its real value we will have***: $\boldsymbol{d\pi = r(v - \Delta S)dt}$ the PDE equation of the Black Scholes will finally be obtained by replacing $d\boldsymbol{\pi}$ by its derivative values:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\delta^2 S^2 \frac{\partial^2 V}{\partial S^2})dt = (rV - r\frac{\partial V}{\partial S}S)dt$$

$= \frac{\partial V}{\partial t} + \frac{1}{2}\delta^2 S^{2\frac{\partial^2 V}{\partial S^2}} + rS\frac{\partial V}{\partial S} - rV$ (4.4) (the partial differential equation of the Black Scholes model)

After the transformation of the partial differential equation the black schole model for options pricing can be described as the following for both put and calls:

C $= \boldsymbol{SN(d_1) - N(d_2)K\, e^{-r(T-t)}}$ (4.5)

For puts options: $\boldsymbol{P = N(d_2)ke^{-r(T-t)} - SN(d_1)}$ (4.6)

With:

$$\boldsymbol{d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}}$$. And

$$\boldsymbol{d_2 = d_1 - \sigma\sqrt{T-t}}$$

Where:

- S is the stock price.
- K is the option strike price.
- T – t is the time to maturity).
- r is the risk free of interest rate.
- ln is the natural logarithm.
- $\boldsymbol{\sigma}$ the volatilityof the stock.

For negative rates pricing the simple Black Scholes model can’t be used since it doesn’t allow negative inputs of interest rates unless a shift parameter is added to the model like it will be the case later in this study.

The Black Scholes model has however some limits even though it's has been for a long time the model most used on the market. The Black Scholes and its hedging of options contracts constitutes the foundation of modern finance. However:

- The delta hedging process of the Black Scholes should be a continuous process but in real life it's a discrete process hence a hedged portfolio needs to be updated once a week or so on (Grzelak&Oosterlee, 2019).

-Empirical studies of financial series have revealed that the normality assumption for asset prices can not capture heavy tails and asymmetries, present in log-asset returns in practrice.

-The volatility in Black Scholes is an uknknown deterministic function, which is regarded as inconsistent since the inversion of the numerical function of the Black Scholes equation based on market prices extracted from different strikes and fixed maturity produces a so called volatility skew and smile.

2.6.6.1. The Binomial model

The binomial model is one of the methods or model mostly used to value an american type of option. The binomial model operates by using an iterative approach employing multiple periods of time. With that model there are always two possible outcome: a up move or a down move that follows a binomial tree.

-the model's dynamics

Let's denote the current price option price by S. Two price movement can be observed: either the price goes up (S+) or down (S-). The up and down factors can be represented the following way: $\mu=\frac{S^{+}}{S}$, $d=\frac{S^{-}}{S}$

The call option at the exercise price Px will be:

C^{+}= max (0,µS- Px) for an upward movement

C^{-}= max (0, ds-Px) for a downward movement.

The put options for an upward and a downward movements are obtained with:

P^{+}= max(Px-µ, S,0)   and P^{-}=max (Px- ds,0).

There are two types of binomial tree: the fast one and the slow one. In practice, let's try to simulate both of the solutions with the following datas: S0 = 100 # initial stock price

K = 100 # strike price

T = 1 # time to maturity in years

r = 0.06 # annual risk-free rate

N = 3 # number of time steps

u = 1.1 # up-factor in binomial models

d = 1/u

The put value option under the american slow tree equals: 4.654588754602527.

The same put value under the fast american tree equals: 4.654588754602527.

Both of the two solutions give the same put price. However under negative rates the binomial model isn't handy to use since for this study, it's mostly used for american type of payoff.

2.6.6.3.The Heston model

Developed by Stephen Heston, the model is supposed to be an ameliorated version of the Black Scholes model. Heston came up with a mathematical model which kept the volatility as an unpredictable value and follow a random process it's derivation can be done this way (Heston 1990):

First the stock price follows a geometric brownian motion:

dSt= $\boldsymbol{\mu St + \sqrt{Vt}StdZ1}$ (4.7)

dVt= K$(\boldsymbol{\theta - Vt})\boldsymbol{dt + \sigma\sqrt{vt}\, dZ2}$.

-the volatility is a stochastic process

-the variance is a square root diffusion. The two brownian are assumed to be correlated

dZ1, dZ2= $\boldsymbol{\rho dt}$ the return have high peak and fat tails compared to the normal distribbution. The square root is used to avoid negative variance dBt= rBdt

Let's say we want to derive a call option: V(t,St,vt,r,K,T)= V(t,St, vt), V(t,s,v). The Heston model allows the following assumptions:

Two sources of randomness

-incomplete market (there's no arbitrage)

- Delta and sigma hedging

- Many martingale measures.

With a two dimensional Ito's lemma the derivation will be :

dV= $\frac{\partial v}{\partial t}$dt+$\frac{\partial V}{\partial S}\boldsymbol{ds} + \frac{1}{2}\frac{\sigma^2 V}{\partial V^2}\boldsymbol{dV^2} + \frac{\sigma^2 Vy}{\partial v \sigma S}\boldsymbol{dVdS}$ (4.8). By substitution in the original equation

$= \frac{\partial V}{\partial t}\boldsymbol{dt} + \frac{\partial V}{\partial S}\boldsymbol{ds} + \frac{1\partial^2 V}{2\partial S^2}\boldsymbol{vs^2dt} + \frac{\partial V}{\partial v}\boldsymbol{dv} + \frac{1\partial^2 V}{2\partial V^2}\boldsymbol{\sigma^2 V} + \frac{\partial^2 V}{\partial V \partial S}\boldsymbol{\rho\sigma v S dt}$ (4.9)

By combining dt terms:

$(\frac{\partial V}{\partial t} + \frac{1}{2}\boldsymbol{VS^2}\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\boldsymbol{\sigma^2 V}\frac{\partial^2 V}{\partial V^2} + \boldsymbol{\rho\sigma v}\frac{\partial^2 V}{\partial V \partial S})\boldsymbol{dt} + \frac{\partial V}{\partial S}\boldsymbol{dS} + \frac{\partial V}{\partial v}\boldsymbol{dV}$. (4.10)

Let's replace the terms in the bracket by L we will have (LU)(t,s,v)dt+$\frac{\partial V}{\partial S}\boldsymbol{ds} + \frac{\partial V}{\partial v}\boldsymbol{dv}$ for V= V(t,St,Vt,r,K,T). In order to remove the sources of randomness let's first consider:

V=V(t, St,Vt,r,K,T1)

U= (t, St, Vt, r, K, T2) with T2 superior to T1 in the case of both these two options.

DU= (LU) (t, s,v)dt + $\frac{\partial V}{\partial S}\boldsymbol{dS} + \frac{\partial U}{\partial V}\boldsymbol{dV}$. The following parameters are incorporated in order to remove randomness: V = $\Delta\boldsymbol{S} + \boldsymbol{\varepsilon U} + \boldsymbol{\sigma B}$

dV= $\Delta\boldsymbol{dS} + \boldsymbol{\varepsilon dU} + \boldsymbol{\sigma dB}$ with dB=rBdt. By replacing the terms we will have:

$\Delta\boldsymbol{dS} + \boldsymbol{\varepsilon}(\boldsymbol{LU}(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{v})\boldsymbol{dt} + \frac{\partial U}{\partial S}\boldsymbol{dS} + \frac{\partial U}{\partial V}$dV)+$\boldsymbol{\sigma r B dt}$

$=\boldsymbol{\varepsilon}(\boldsymbol{LU})(\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{v})\boldsymbol{dt} + \left(\Delta + \boldsymbol{\varepsilon}\frac{\partial V}{\partial S}\right)\boldsymbol{ds} + \boldsymbol{\varepsilon}\frac{\partial U}{\partial V}\boldsymbol{dv} + \boldsymbol{\sigma r B dt}$ to remove ds and dt we will proceed by:

$$\frac{\partial V}{\partial S} = \Delta + \boldsymbol{\varepsilon}\frac{\partial U}{\partial S} \quad \boldsymbol{in\ terms\ of\ delta\ it\ will\ be}\ \Delta = \frac{\partial V}{\partial S} - \boldsymbol{\varepsilon}\frac{\partial U}{\partial S}$$

$$\frac{\partial V}{\partial v} = \boldsymbol{\varepsilon}\frac{\partial U}{\partial v}\ \boldsymbol{in\ terms\ of\ \varepsilon\ it\ willbe\ \varepsilon} = \frac{\partial V}{\partial v} / \frac{\partial U}{\partial v}$$

By substitution we will then have: $\boldsymbol{\varepsilon(LU)(t,s,v)dt} + \left(\frac{\partial V}{\partial S} - \boldsymbol{\varepsilon}\frac{\partial U}{\partial S} + \boldsymbol{\varepsilon}\frac{\partial U}{\partial S}\right)\boldsymbol{ds} + \frac{\partial V}{\partial v}/\frac{\partial U}{\partial v}\frac{\partial U}{\partial v}\boldsymbol{dv} + \boldsymbol{\sigma rBdt}$ (4.11)

By simplification: $\boldsymbol{\varepsilon(LU)(t,s,v)dt} + \frac{\partial V}{\partial S}\boldsymbol{dS} + \frac{\partial V}{\partial S}\boldsymbol{dv} + \boldsymbol{\sigma rBdt}$ with the initial equation we can spot matching terms:

LV(t,s,v)dt+$\frac{\partial V}{\partial S}\boldsymbol{ds} + \frac{\partial U}{\partial v}\boldsymbol{dv} = \boldsymbol{\varepsilon(LU)(t,s,v)dt} + \frac{\partial V}{\partial S}\boldsymbol{ds} + \frac{\partial V}{\partial v}\boldsymbol{dv} + \boldsymbol{\sigma rBdt}$ we will be left with only deterministic terms:$\boldsymbol{LV(t,s,v)dt = \varepsilon(LU)(t,s,v)dt + \sigma rBdt}$

V=$\Delta\boldsymbol{S} + \boldsymbol{\varepsilon U} + \boldsymbol{\sigma}$

$$\boldsymbol{\sigma B = V - \Delta S - \varepsilon U}$$

$$= \text{V-}(\frac{\partial V}{\partial S} - \boldsymbol{\varepsilon}\frac{\partial U}{\partial S})\boldsymbol{S} - \boldsymbol{\varepsilon U}$$

$$= \text{V-}\frac{\partial U}{\partial S}\boldsymbol{S} + \boldsymbol{\varepsilon}\frac{\partial U}{\partial S}\boldsymbol{S} - \boldsymbol{\varepsilon U}$$

We can substitute the expression for $\boldsymbol{\sigma B}$ into the main expression

LV$\boldsymbol{(t,s,v)dt - \varepsilon(LU)(t,s,v)dt + r}\left(\boldsymbol{V} - \frac{\partial V}{\partial S}\boldsymbol{s} + \boldsymbol{\varepsilon}\frac{\partial U}{\partial S}\boldsymbol{s} - \boldsymbol{\varepsilon U}\right)\boldsymbol{dt\ by\ shifting\ the\ terms}$:

(LV) (t,s,v)-rV+$\frac{\partial V}{\partial S}\boldsymbol{s} = \boldsymbol{\varepsilon(LU)(t,s,v) + r\varepsilon}\frac{\partial U}{\partial S}\boldsymbol{S} - \boldsymbol{r\varepsilon u}$ by factoring $\boldsymbol{\varepsilon = \varepsilon(LU)(t,s,v) - rU + r}\frac{\partial U}{\partial S}\boldsymbol{S} - \boldsymbol{r\varepsilon U}$ by factoring $\boldsymbol{\varepsilon}$ we will have = (LV)(t,s,v)-rv+$\boldsymbol{r}\frac{\partial V}{\partial S}\boldsymbol{S} = \frac{\partial V}{\partial v}/\frac{\partial U}{\partial v}$

((LU)(t,s,v)-rU+$\boldsymbol{r}\frac{\partial U}{\partial S}\boldsymbol{S}) = ((\boldsymbol{LV}),(\boldsymbol{t,s,v}) - \boldsymbol{rV} + \boldsymbol{r}\frac{\partial V}{\partial S}\boldsymbol{S})/\frac{\partial V}{\partial v} = ((\boldsymbol{LU})(\boldsymbol{t,s,v}) - \boldsymbol{rU} + \boldsymbol{r}\frac{\partial U}{\partial S}\boldsymbol{S})/\frac{\partial U}{\partial v}$

The fraction here must not depend on V and U but it certainly depends on the parameter, here we will ignore the constant:

((LV)(t,s,v)-rV+r$\frac{\partial V}{\partial S}\boldsymbol{S})/\frac{\partial V}{\partial v}$ =-F(t,s,v) by shifitng the denominator

(LU)(t,s,v)-$\boldsymbol{rU + rs}\frac{\partial V}{\partial S} = \boldsymbol{-F(t,s,v)}\frac{\partial V}{\partial v}$

(LV)(t,S,V)-$rv = -rS\frac{\partial V}{\partial S} - F(t,s,v)\frac{\partial V}{\partial v}$ rs is the drift of the stock price F must be a sort of the drift of the variance. It's an arbitrary fonction ds=$usdt + \sqrt{vs}dz1$

The variance given by dv=$K(\theta - V)dt + \sigma\sqrt{t}dZ2$

ds= $rsdt + \sqrt{v}\, sdZ1^{Q}$, $Z1^{Q}$ *is the brownian motion under risk neutral Q*

ds =($\frac{u - \frac{u-r}{\sqrt{v}}\sqrt{v}}{r}$)$dt + \sqrt{v}\, dZt^{Q}$

The gaussian theorem under the risk dynamics of the CAPM helps to determine the equation of dv

dv=(K($\theta - v$) $- \tau\sigma\sqrt{v}$) $dt + \sigma\sqrt{v}\, Z2^{Q}$. Here the volatility is not a traded asset we can't write the drift as equal to rs hence this equation is the equevalent of rs

F(t,s,v)= K($K - \theta$)$\tau\sigma\sqrt{v} = -rs\frac{\partial V}{\partial S} - (K(\theta - v) - \tau\sigma\sqrt{v})\frac{\partial V}{\partial v}$

Let's recall L= $\frac{\partial}{\partial t} + \frac{1}{2}\frac{vS^2\partial^2}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2} + \rho\sigma v\frac{\partial^2}{\partial v.\partial S}$ we can substitute and find the PDE of the heston model:

$\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial V^2} + \rho\sigma vS\frac{\partial^2 V}{\partial V\partial S} - rV$ (4.13)

A call option can be obtained by discretizing the exact solution of the heston model. A European option with a different type of payoff or multiple strike prices can be priced by the Heston model. In practice the heston model is often associated with the Euler discretization to obtain the exact solution of an option price.

*European option pricing with the exact solution of the Heston model: Application example.

European call option are computed by means AES with $S(t_0) = 100$ *for* 3 *strike prices* : $K = 100ATM$, $K = 70(ITM)$*and* $K = 140OTM$ In the experiment different time steps are used varying from one stepper year to 64 time steps per year. The model parameters are choosen as: $\kappa = 0.5, y = 1, V_0 = 0.04$, $r = 0.1, \rho x, v = -0.9$. The model's calibration with both Euler and Milstein scheme gives:

EUROPEAN OPTION PRICING

Exact option price = [0.73585362]

For N = 100 Euler scheme yields option price = 0.7285547612550968 and Milstein 0.728444051871542

For N = 100 Euler error = [-0.00729886] and Milstein error [-0.00740957]

For N = 1000 Euler scheme yields option price = 0.7200580888738554 and Milstein 0.7202098357606097

For N = 1000 Euler error = [-0.01579553] and Milstein error [-0.01564379]

For N = 5000 Euler scheme yields option price = 0.7481360160804368 and Milstein 0.7481404588408821

For N = 5000 Euler error = [0.0122824] and Milstein error [0.01228684]

For N = 10000 Euler scheme yields option price = 0.7387184512149807 and Milstein 0.7386727436156081

For N = 10000 Euler error = [0.00286483] and Milstein error [0.00281912]

CASH OR NOTHING PRICING

Exact option price = [[2.44829963]]

For N = 100 Euler scheme yields option price = 2.731117147394321 and Milstein 2.731117147394321

For N = 100 Euler error = [[0.28281751]] and Milstein error [[0.28281751]]

For N = 1000 Euler scheme yields option price = 2.4438789646511254 and Milstein 2.434461319315283

For N = 1000 Euler error = [[-0.00442067]] and Milstein error [[-0.01383832]]

For N = 5000 Euler scheme yields option price = 2.4241019094458562 and Milstein 2.422218380378687

For N = 5000 Euler error = [[-0.02419773]] and Milstein error [[-0.02608125]]

For N = 10000 Euler scheme yields option price = 2.422218380378687 and Milstein 2.422218380378687

For N = 10000 Euler error = [[-0.02608125]] and Milstein error [[-0.02608125]]

2.6.6.4.The SABR model

The SABR model is used as an extension of the Black Scholes model. The model is mostly used as a stochastic volatility model. Unlike the Heston moodel the SABR model doesn't produce prices directly. It mostly insists on the implied volatility that's observed in the Black Scholes model. The SABR model of Hagan is defined by the following 3 equations:(Hagan, 2002):

$$\boldsymbol{dft = \sigma tft^{B} dWt^{1}} \text{ (4.14)}$$

$$\boldsymbol{d\sigma t = v\sigma t dWt^{2}} \text{(4.15)}$$

$$\boldsymbol{E[dWt^{1}\, dWt^{2} = \rho dt} \text{(4.16)}$$

With the initialvalues $\boldsymbol{fo\ and\ \sigma = \sigma o, ft}$ is the forward rate $\boldsymbol{\sigma}$ is the volatility and $\boldsymbol{Wt^{1}}$ and $\boldsymbol{Wt^{2}}$ are correlated brownian motions, with the correlation parameter $\boldsymbol{\rho}$. The parameters are:

$$\boldsymbol{\sigma: the\ variance}$$

$$\boldsymbol{v = the\ volatility\ of\ the\ variace}$$

$$\boldsymbol{B = the\ exponant\ for\ forward\ rate}$$

$$\boldsymbol{\rho = correlation\ between\ the\ brownian\ motion}$$

- SABR implied volatility and option prices

Prices of European type of options in SABR model are given by Black Scholes model for a current forward rate, strike price and implied volatility. The price of a call option at maturity T is:

$$\boldsymbol{CB(f, k, \sigma B, T) = e^{-rt}[fN(d1 - KN(d2)]} \text{ (4.17)}$$

$$\boldsymbol{d1, 2 = \frac{lnf/K \pm \frac{1}{2}\sigma^{2}BT}{\sigma B\sqrt{T}}}$$

And analogously for a European put. The volatility parameter is determined by the SABR model with the estimates of $\boldsymbol{\sigma, v, B, and\ \rho}$ the implied volatility is defined by:

$$\boldsymbol{\sigma(K, f) = \frac{\left\{1 + \left|\left[\frac{1-B}{24}\right)^{2} \frac{\sigma^{2}}{(f,k)^{1-B}} + \frac{1\rho Bv\sigma}{4(f,k)^{1-B)/2}} + \frac{2-3\rho^{2}}{24} v^{2}\right] T\right\}}{(f,k)^{1-B)/2}\left[1+\frac{1-B}{24})^{2} ln^{2} f/k + \frac{(1-B)^{4}}{1920} \ln f/k\right]}} \quad \text{(4.18)}$$

$$\boldsymbol{x \frac{Z}{\chi(z)}}$$

Z=$\frac{\boldsymbol{v}}{\boldsymbol{\sigma}}(\boldsymbol{f},\boldsymbol{k})^{(\mathbf{1}-\boldsymbol{B})^2}\boldsymbol{ln}\frac{\boldsymbol{f}}{\boldsymbol{k}}$

$$\boldsymbol{\chi}(\boldsymbol{z}) = \left[\frac{\sqrt{\mathbf{1}-\mathbf{2}\boldsymbol{\rho}\boldsymbol{Z}+\boldsymbol{Z}^{\mathbf{2}}}+\boldsymbol{Z}-\boldsymbol{\rho}}{\mathbf{1}-\boldsymbol{\rho}}\right]$$

Once the parameters $\boldsymbol{\sigma}$, $\boldsymbol{B}, \boldsymbol{\rho}, \boldsymbol{and}\ \boldsymbol{v}$ are estimated, the implied volatility is the fonction of the forward rate. Since the SABR model determines the implied volatility for a single maturity, the dependence of $\boldsymbol{\sigma B}$ on T is not reflected in the notation $\boldsymbol{\sigma B}(\boldsymbol{k},\boldsymbol{f})$.

2.6.6.5. A pricing approach

Two major approaches can be adopted for pricing an option: A PDE approach can be used or the risk neutral probability. The idea behind a risk neutral probability is to change the probability measure from a true probability to a risk neutral probability. The difference between the two measure is called the expected return or payoff of a stock. In the true probability measure the expected payoff is denoted μ, whereas in the risk neutral probability the stock's expected payoff is a risk free rate "r" (Grzelak&Oosterlee, 2019).

When pricing a derivative both those two measures should be chosen according to the informations that they provide. μ, is considered to be estimated measure obtained from historical data (forward looking and backward looking) and "r" provides some estimations based on market datas. Under the risk neutral measure, market evolution and volatilities are taken into consideration. Further more it also advised to rely on existing underlyings and if there's inconsistency in prices, an arbitrage process is required.

2.7. Options pricing and uncertainty.

When pricing any type of asset, uncertainty is always the main parameter that impacts the return or the price. As a result every pricing model in the finance industry has to incorporate the volatility parameter in its components. For options pricing, there's implied volatility parameters, jump diffusions based model (abnormal behaviors in prices) and stochastic volatility models.

2.7.1. Implied volatility.

Risk is a major factor for options, therefore under normal and allowed circumstances the option's value is exponentially an increasing function of the volatility. This implies that there's a strong relationship between an option price and its associated volatility. To put it into words as rebonatto stated in 1999, the implied volatility is the wrong number in the wrong formula to get the right price. Mathematically it can be defined by the following expression: $\mathrm{V}\boldsymbol{c}(\boldsymbol{t},\boldsymbol{s}) = \boldsymbol{BS}(\boldsymbol{\sigma};\boldsymbol{r};\boldsymbol{T};\boldsymbol{K};\boldsymbol{S_0})$

Where Bs is constanly increasing in $\boldsymbol{\sigma}$ hence higher volatility always correspond to highere prices. Now let's assume the existence of some inverse function.

$$\boldsymbol{g\sigma}(*) = \boldsymbol{BS^{-1}}(*)$$

$$\boldsymbol{\sigma impl} = \boldsymbol{g\sigma}(\boldsymbol{V_c^{mkt}};\boldsymbol{r};\boldsymbol{T};\boldsymbol{K};\boldsymbol{S_0})$$

*How to find then the implied volatility?

There are so many solutions to solve the inverse equation of the Black Scholes but the most commonly used for the implied volatility is the Newton and Brent solution. Since underlyings prices can vary quickly it's important to use the most efficient method when calculating implied volatilities.

*The Newton Method

Let's consider an approximation $\boldsymbol{Xn}$. Assume that "g" is differentiable and write: $\in_{\boldsymbol{n}} = \boldsymbol{Xex} - \boldsymbol{Xna}$ (the first term of the equation is the exact solution, the seconde term is called the iteration). The taylor series expansion gives:

$$0=\boldsymbol{g}(\boldsymbol{Xex}) + \in_{\boldsymbol{n}} \boldsymbol{g}'(\boldsymbol{Xn}) + \frac{\in_n^2}{2}\boldsymbol{g}''(\boldsymbol{Xn}) + \cdots \text{ (4.19)}$$

Ignoring the terms of the second order we will have $\boldsymbol{Xex} \approx \boldsymbol{Xn} - \frac{g(Xn)}{g'(Xn)}$ the following expression used as our new approximation we will have: $\boldsymbol{Xn}+\boldsymbol{1} = \boldsymbol{Xn} - \frac{g(Xn)}{g'(Xn)}$. How fast then does the error reduce with this approximation? With one additional term in the taylor series we had before we get:

$\in_{n+1=} \boldsymbol{Xex} - \boldsymbol{Xn} + \boldsymbol{1} = \in_{\boldsymbol{n}} \frac{g(Xn)-g(Xex)}{g'(Xn)} \approx -\in_n^2 \frac{g''(Xn)}{2g'(Xn)} (\boldsymbol{estimated\ error})$. Now let's suppose : $\mathrm{g}[\mathrm{a,b}] \to \mathbb{R}\ \boldsymbol{is\ a\ differentiable\ function\ .From\ basic\ calculus\ we\ have}$: $\boldsymbol{g}'(\boldsymbol{Xn}) = \frac{g(Xn)-0}{Xn-Xn+1} = \frac{0-g(Xn)}{Xn+1-Xn}, \boldsymbol{n} = \boldsymbol{0}, \boldsymbol{1}$; the following iteration is then obtained:

$\boldsymbol{Xn+1 = Xn - \frac{g(Xn)'}{g'(Xn)}, n = 0,1 \; here\ the\ term\ g(Xn)'can\ be\ regarded\ as\ g(\sigma) =}$ $\boldsymbol{BS(\sigma) - V_c\, mkt}$. With an arbitrary initial value X_0, In the case of the Black Scholes pricing equation we can hedge vega which is the sensibility of the Black Scholes model to the volatility:

$$\boldsymbol{\sigma n + 1 = \sigma n - \frac{BS(\sigma n;*) - V_c mkt}{\frac{\partial BS(\sigma n;*)}{\partial \sigma n}}} \quad (4.20)$$

As an example, let's consider the following parameters:

V_market = 2 market call option price

K = 120 strike

tau = 1 time-to-maturity

r = 0.05 # interest rate

S_0 = 100 today's stock price

SigmaInit = 0.25 Initial implied volatility

CP ="c" C is call and P is put

The implied volatility obtained after calibration becomes:

Iteration 1.0 with error = 3.025413481792615

Iteration 2.0 with error = 0.19134998568795325

Iteration 3.0 with error = 0.0022254541477302325

Iteration 4.0 with error = 3.353154056640051'e-07

Iteration 5.0 with error = 1.0658141036401503'e-14

Implied volatility for CallPrice= 2, strike K=120,

Maturity T= 1, interest rate r= 0.05 and initial stock S_0=100

Equals to sigma_imp = 0.1614827

Option Price for implied volatility of 0.1614827288413938 is equal to 2.0

2.7.2. Jump processes.

Jump diffusion and Lévy based models are quite captivating since they explain the jump patterens manifested by some stocks. Jumps in stocks returns have been observed in the market especially periods of financial crisis like in 1987, 2000 or 2008. Jumps model are more realistic when pricing underlyings closer to maturity time.

Jump process models are superior to the Black Scholes in the sense that daily log returns have heavy tails and for longer periods of time jump processes approach normality which is consinstent with empirical studies.

Empirical densities are often considered peaked compared to the normal density. A phenomenon known as excess Kurtosis. By inserting some parameters one control the kurtosis and the so called asymetry of the log return density is able to fit in the smile curve of the implied volatility (Grzelak&Oosterlee, 2019).

***Jump diffusion process.**

Let's extend the black model by independent jumps driven by a poisson process:

$X(t) = \mu dt + \sigma dW^p(t) + J dX_p(t)$.(4.21)

Where μ is the drift, σ the volatility parameter, $X_p(t)$ a poisson process and J gives the jump magnitude deducted from the distribution F_J . $W^p(t)$ and $X_p(t)$ are assumed to be independent.

a) *Definition of a poisson random variable.*

Xp counts the number of occurences of an event during a given period of time. Probability of observing K≥ 0 *occurence in a time period.* $\mathbb{P}[X_P = K] = \frac{\sum_p^k e^{-\mathcal{E}p}}{K!}$;

E[X_P]= $\mathcal{E}p$(average number of occurence; Var[X_P]= $\mathcal{E}p$

b) *Poisson process*

$X_p(t); t \geq t_0 = 0$; *with* $\mathcal{E}p > 0$ *is an integer valued as a stochastic process* . $X_P(0) = 0$

$t_0 = 0 < t1 < \cdots < tn$. the increments $X_P(t_1) - X_P(t_0), X_P(t_2) - X_P(t_1) \ldots . X_P(t_n) - X_P(t_n - 1)$ are independent random variable for S$\geq 0, t > 0$ *and* $K \geq 0$, increments have a poisson distribution. $\mathbb{P}\left[X_p(S + t) - X_p = K\right] = \frac{(\mathcal{E}_P t) * e^{-\mathcal{E}pt}}{K!}$

The probability of one event occuring in a small time interval dt is given by:

$$\mathbb{P}\left[X_p(s + dt) - X_p(s) = 1\right] = \frac{(\mathcal{E}_p dt)e^{-\mathcal{E}_p dt}}{1!} = \mathcal{E}_p dt + 0(dt) \quad (4.21)$$

The opposite can be defined by:

$$\mathbb{P}\left[X_p\,(S + dt) - X_p(S) = 0\right] = e^{-\mathcal{E}_p dt} = 1 - \mathcal{E}_p\, dt + 0(dt) \quad (4.22)$$

The simulation of a poisson shows the following results for an asset’s price:

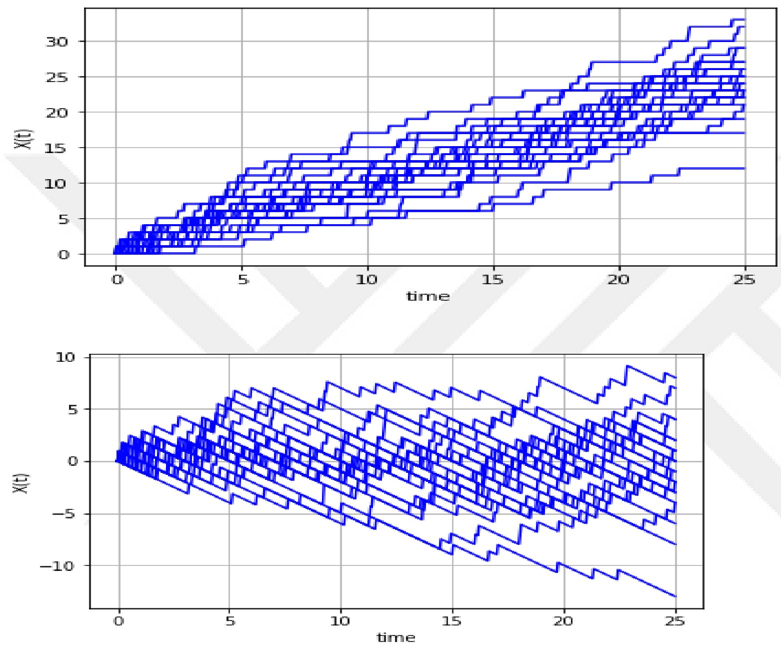


Figure 6: Paths under jump-diffusion with a Poisson random variable (1000)

2.7.3.Stochastic volatility.

Modelling volatility as a random variable is confirmed by pratical data that indicate the variable and impredictable nature of volatility (Hull and White, Stein and Stein, 2002; Heston, 1900; Schöbel and Zhu, 1989).

The variance process resulting in any a stochastic differential equation can be recognised as mean reverting square- root process, originally the process was introduced by Cox-Ingersoll in 1985, to model the spot interest rate. If the variance surpasses its mean; it can be driven back to the mean with the speed of the mean reversion

- *Stochastic volatility processes example of the CIR model*

1) the variance process of the CIR is is a stochastic process defined by:

$$\boldsymbol{dV(t) = k\left(\overline{V} - V(t)\right)dt + \lambda\sqrt{V(t)}dW(t)} \text{ .(4.23)}$$

This equation is also regarded as the variance part of the Heston model.

2) For a time t$> \mathbf{0}$, the varaince V(t) is normally distributed as $\overline{\boldsymbol{c}}(\boldsymbol{t})$ times a non central Chi-squared random variable $\boldsymbol{\chi^2(\overline{d}, \overline{\lambda}\,(t) with\, \overline{d}}$ the degrees of freedom parameter $\overline{\boldsymbol{\lambda}}\,(\boldsymbol{t})$.

In practice we will have: $\boldsymbol{V(t) \sim \overline{c}\chi^2(\overline{d\lambda}(t), t > 0}$ in this equation $\boldsymbol{\chi^2}$ is a non central chi squared distribution, $\overline{\boldsymbol{\lambda}}(\boldsymbol{t})$ is a no centrality parameter at point "t"

With $\boldsymbol{\overline{c}(t) = \frac{1}{4k}\Upsilon^2\left(1 - e^{-kt}\right), d = \frac{4k\overline{V}}{\Upsilon^2}, \overline{\lambda}(t) = \frac{4k\overline{V_0}e^{-kt}}{\Upsilon^2(1-e^{-kt})}}$ (4.24)

4) The square root process for the variance avoids negative values for V(t) and if V(t) reaches zero it will afterwards becomes reaches zero. It's the feller condition: $\boldsymbol{2k\overline{V}} \geq \Upsilon^{\mathbf{2}}(\boldsymbol{volatility\ squared})$ which guarantees that V(t) stays positive. On the other hand if the feller condition is not satisfied the variance processhase huge probabilities of reaching zero.

Let's try to simulate a stock under the cox ingersoll stochastic process given the following parameters:

NoOfPaths = 1

NoOfSteps = 2000

T = 1

Kappa =0.7

V0 =0.1

Vbar =0.1

Gamma =0.7

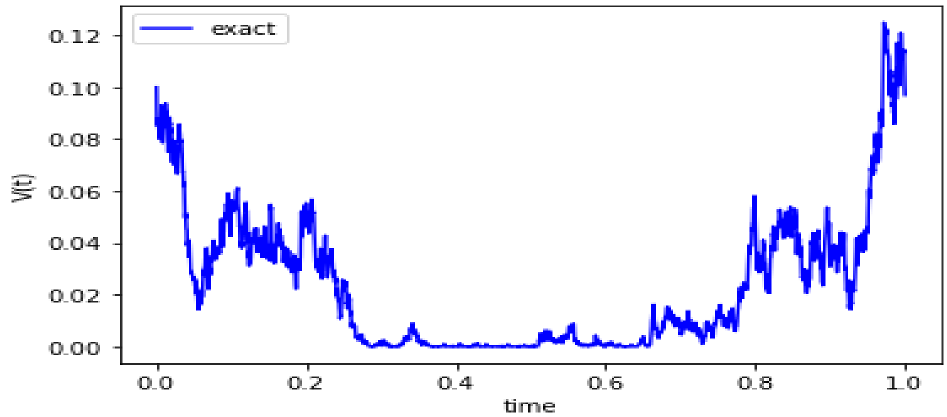


Figure 7: Volatility smile under the stochastic volatility mode of Heston

2.7.4.How to choose a pricing method?

A pricing approach always start with a certain type of financial product. The next step should be the modelling of product prices correspondingly, on this step preferential differential equations of a model are introduced. Once this step is completed, a calibration of the market data is conducted. On this step an analyst proceed by optimization and hedging probable risks (for derivatives the greeks are applied to a pricing formula). Pricing is often resumed into following steps (Oosteerlee, 2019):

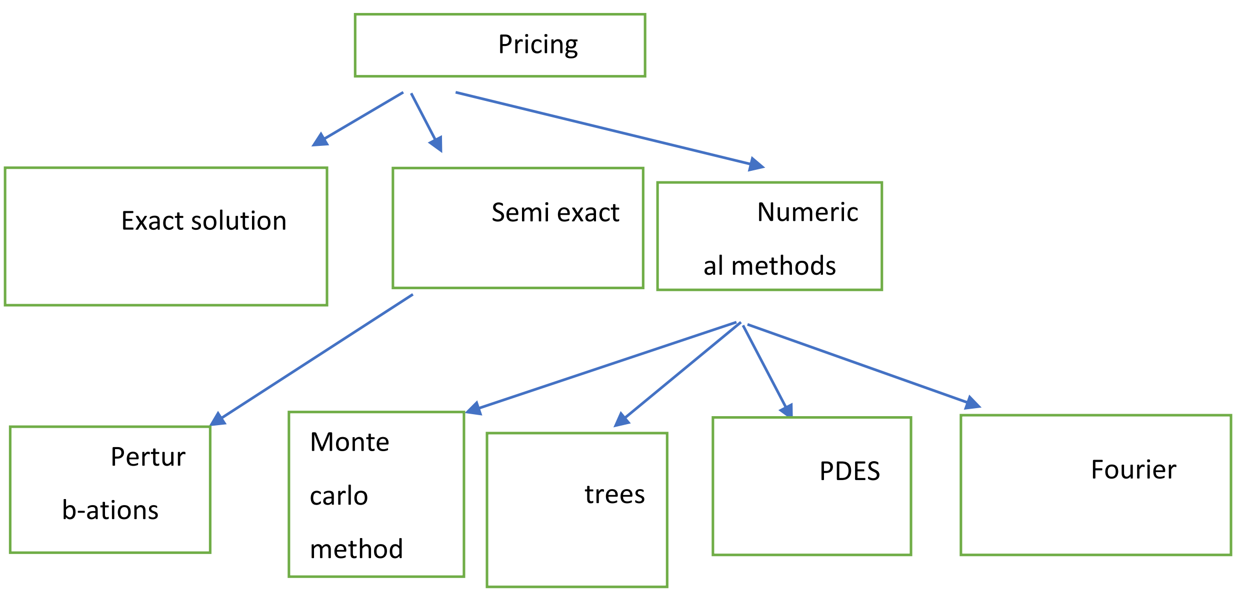


Figure 8: Approach to choosing a pricing method (Oosterlee, 2019)

The monte carlo simulation is often multi-dimensional, PDE based models are expensive but have an advantage of risk hedging based on the Greeks and the fast fourier based methods are are very fast for the computation of simple payoffs which makes them not reliable for exotic options pricing.

2.8. Advanced pricing methods for European options

Often when pricing, model that are fast to compute with accurate results and less errors are necessary. Hence the use of the fast fourier transform and the montecarlo simulation. However those models are used for simple payoffs and need to be constantly updated.

2.8.1.Motivation behind the fourier transform:

The fast fourier methods derive pricing methods that:

- Are considered fast to compute
- Are restricted to gaussian based models
- Should work as long as we have the characteristic fuction:

$$\phi(\mu)=\int_{-\infty}^{\infty} \boldsymbol{e^{i\mu x}}\, \boldsymbol{F(x)dx}. \quad (5.0)$$

In probability theory, a characteristic function of a continuous process of a random variable “x” equals the fourier transform of the density of x. The fourier transform is commonly used for simple payoffs and can easily be used to price European type of options. In order to obtain a characteristic function, the density needs to be integrated. (Grzelak&Oosterlee, 2019)

Let’s assume we have a function F defined by: $\mathbb{R}\rightarrow \mathbb{R}$ $\boldsymbol{which\ is\ in\ L_1, ie,} \int_{-\infty}^{+\infty} |\boldsymbol{F(x)}|\boldsymbol{dx} < \infty$. $\boldsymbol{F(x)}$ is regarded as a continuous function. The fast fourier transform of the function F is defined as: $\phi_{\boldsymbol{x}}\ (\mu) = \mathbb{E}[\boldsymbol{e^{i\mu x}}] = \int_{-\infty}^{+\infty} \boldsymbol{e^{i\mu x} F(x)dx} = \int_{-\infty}^{+\infty} \boldsymbol{e^{i\mu x}}\, \boldsymbol{dF(x)}$. (5.1)

$\phi_{\boldsymbol{x}}\ (\mu)$ is a characteristic function, $\boldsymbol{F(x)}$ is density function defined by: $\boldsymbol{\frac{df}{dx}}$, and X∈ $\mathbb{R}$

**** Characteristic function and useful properties.***

The most important fact regarding ϕ_x (μ) is that it uniquely determines the distribution function of "X" In addition to that, the moments of random variable "X" can also be derived by ϕ_x (μ) as $\mathbb{E}[X^k] = \frac{1 d^k}{i^k\, d\mu^k} \phi_x(\mu)|_{\mu=0}$ (derivative of a characteristic function) (5.2)

-Fourier cosine expansions for European options.

Let's consider the risk neutral valuation formula: $V(X, t_0) = e^{-r\Delta t}$ by adding the expectation term under the risk neutral valuation we have $\mathbb{E}^Q[V(y,T)|X] = e^{-r\Delta t}$. Under integration a payoff of a derivative V(x,t) and the density F(x) can also be interpreted like $\int_0^{\mathbb{R}} V(y,T)F(y|X)dy$ The payoff for European options after the adjustment of a log- asset price in practice becomes:

$$Y(T) = \log\left(\frac{S(T)}{K}\right) by\ substitution\ V(T,y) = [\overline{\alpha}.K(e^y - 1)] with \overline{\alpha} \begin{cases} 1 for\ a\ call \\ -1\ for\ a\ put \end{cases}$$

By focusing on a call option in the case where $a < 0 < b$ *we obtain* : $H_K call = \frac{2}{b-a}\int_0^b K(e^y - 1)\cos(K\pi\frac{y-a}{b-a})\,dy = \frac{2}{b-a}K(\chi_k(0,b) - \psi(0,b))$ (5.3)

Simulary, we can find a price for a put option $H_k put = \frac{2}{b-a}K(-\chi_k(a,0) + \psi(a,0)$. If we try an application example for the cosine methods of an option with the following parameters: CP = "c"

S0 = 100.0

r = 0.1

tau = 0.2

sigma = 0.40

K = [85.0,90.0,100.0,110.0]

N = 4*32

L = 1

The following result is obtained:

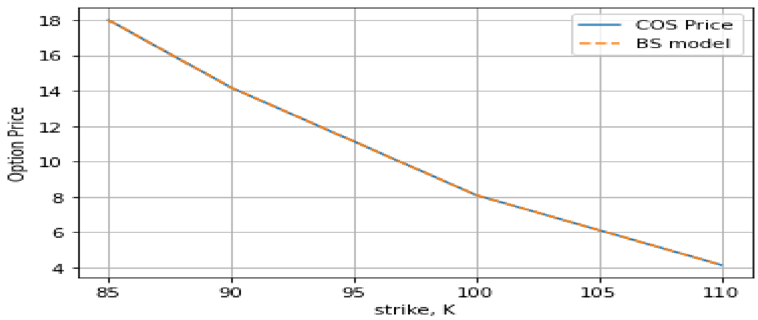


Figure 9: comparison of the Black-Scholes and the Cosine model

Here the graph compare the option price obtained from the cosine method and the Black Scholes model. The two models converge perfectly on the same price. The fourier transform models are reputed to give faster results like it's going to be illustrated in the following example:

Let's assume the following parmeters after calling numpy and matplotlib in python:

The risk free rate = 0.05, the initial stock price :200 with an ATM strike: 200, the sigma:0.2, N paramters:(40,60,80,120,140) L= 6 the following results are obtained:

Reference value is equal to =98.44271472257104

For N=40 the error is = [0.04544434]

It took 0.0001924901008605957 seconds to price.

For N=60 the error is = [0.00210211]

It took 0.0002273871898651123 seconds to price.

For N=80 the error is = [3.49496057'e-05]

It took 0.0002239844799041748 seconds to price.

For N=100 the error is = [2.03506076'e-07]

It took 0.00023337435722351073 seconds to price.

For N=120 the error is = [4.08888923'e-10]

It took 0.0002842404842376709 seconds to price.

For N=140 the error is =[2.7000624e-13]

It took 0.0002623107433319092 seconds to price.

2.8.2.The Montecarlo simulation

The montecarlo simulation is often associated with some statistics samples. The integration part of the montecarlo simulation is a sampling techinic based on probability theory. When revealing informations the methods rely on trials. From a perspective point of vue, montecarlo methods rest on the cental limit theorem and the law of large number. The advantage of using statistical sampling methods based on montecarlo simulation is that they are able to solve quite large complicated problems. Since the results of an experiment is a random number, the error from simulations has a probabilisitc distribution (Grzelak, 2019).

The Montecarlo algorithm has the following assumptions:

1. Partition the time interval [0,T] ; 0=$t_0 < t_1 < \cdots < t_m = T$
2. Provide asset values $S_i j$ using the risk neutral probability of a given underlying model, $S_i j$ has two indices: the time point and the montecarlo paths
3. Calibrates N payoff values; H_j; in the case of a vanilla option. $H_j = H(T; S_m j$ In the case of path dependent potions $H_j = H(T; S_i j); i = 1, \ldots..m$
4. Easily calibrate the average $\mathbb{E}^Q[H(T,S)|F(t_0)] \approx \frac{1}{N}\sum_{j=1}^{N} H_j = H_N$
5. Calculate an underlying value as $V(t_0, S) \approx e^{-r(T-t_0)} \frac{1}{N}\sum_{j=1}^{N} H_j$
6. Is able to determine the distribution of a standard error obtained in the pricing process (step)

**** Standard error for the montecarlo simulation.***

By considering the law of large number we know that for N→ ∞, Lim$\overline{H}(T,S) = \mathbb{E}^Q[H(T,S)]$ with probability 1 for estimated error due to calibrated finite number of paths. $Var^Q[\overline{H}N(T,s)] = Var^Q[\frac{1}{N}\sum_{j=1}^{N} H(T, S_m j)]$ (5.4)

The first term of this equation is the variance of the mean. From the last equation we get the following expression for the variance $Var^Q[\frac{1}{N}\sum_{j=1}^{N} H(T, S_m j)] = \frac{1}{N^2}\sum_{j=1}^{N} H(T, smj)] \approx$

$\frac{1}{N} Var^Q[H(T,S]$ given the fact that sample Smj are drawn independently. The unknown variance is approximated by: $\overline{v^2} = \frac{1}{N-1}\sum_{j=1}^{N}(H(T,Smj) - \overline{H_N}((T,S))^2$. The standard error $\varepsilon n \text{ is defined by } \varepsilon n = \frac{\overline{vN}}{\sqrt{N}}$. (5.5)

For its application, let's consider the following example:

S = 100.89

K = 98.01

vol = 0.091

r = 0.015

N = 10 #number of time steps

M = 1000 # number of simulations

Market_value = 3.86

T = 3.0.

The market value of an option has the following value:

Call value is $10.18 with SE +/- 0.16

The Montecarlo handles large number and it's computation can be sometimes erroneous due t-o the standard error feature.

Interest rates products.

2.9.Interest rates products.

Since we are trying to study how options or derivatives behave under negative interest rates, it's important to understand some interest rates products and the dynamics behind their pricing. Here's a list of some interest rates products commonly observed on the market (Grzelak&Oosterlee, 2019):

2.9.1.Simple compounded forward rate.

Let's consider two counter parties A and B. The counter party A have to pay counter party B 1€ at time T1 and at time T2 counter party will receive back 1€ with the interest rate "K". The fair value of this agreement is mathematically represented the following way:

$$V(t_0) = \mathbb{E}^Q[\left(\frac{1}{M(T_1)} + \frac{1+K(T_2-T_1)}{M(T_2)}\middle| f(t_0)\right] = -P(t_0, T_1) + (1 + (T_2 - T_1)K)P(t_0, T_2) \quad (5.6)$$

The second part of the equation is obtained after epxressing the interest rate "K" by its real value defined by the following expression: $K = \frac{1}{(T_2 - T_1)}(\frac{P(t_0, T_1)}{P(t_{0,}T_2)} - 1)$ (5.7)

Generally the rate for interbank lending starts at the date Tk-1 and maturity Tk with tenor $T_k = T_k - T_k - 1$ dnoteed by $K = l_k(t) := l(t: T_k - 1, T_k)$ also equal to the libor rate which is defined by

$$l(\ t; T_k - 1\ , T_k) = \frac{1}{T_k}(\frac{P(t, T_k - 1}{P(t, T_k)} - 1 \quad (5.8)$$

2.9.2.The forward rate agreement.

In the interest rate market, an interest rate at future time $T_k - 1$ can be fixed. For an accrued period of time $[T_k - 1, T_k]$ two parties can agree to Exchange a fixed rate "K" for a payment of the (floating) libor rate observed at time $(T_k - 1)$. The payoff of the FRA is given by the following relationship: $V^{FRA}(T_k - 1) = H^{FRA}(T_k - 1) = \frac{T_K(l(T_K - 1; T_k - 1; T_k) - k}{1 + T_K l(T_k - 1; T_k - 1, T_k)}$. If we consider the definition of the libor rate, we can immediately connect the denominator part with the zero coupon bond $P(T_k - 1, T_k)$ the following way: $P(T_k - 1, T_K) = \frac{1}{1 + T_k l(T_k - 1\ ; T_k - 1, T_k)}$ In order to obtain the following expression for the FRA's payoff : $V^{FRA}(T_k - 1) = T_k P(T_k - 1; T_k)(l, T_k - 1; T_k - 1, T_k) - k)$. (5.9)

The payoff can be also be expressed under the real World measure "Q" by the following equation: $\mathbb{E}^Q[\frac{1}{M(\ T_k - 1)} T_k P(T_k - 1; T_k)(l(T_k - 1\ , T_k)(l(T_k - 1; T_k - 1\ T_k) - k)|\ f(t_0)]$.Since the bonds $P(T_k - 1; T_k)$ are considered to be a traded asset, discounted bonds should therefore be martingales which give us the following relation for a FRA: $V^{FRA}(t_0) = P(t_0, T_k - 1\ ) - P(t_0, T_k) - T_k K P(t_0, T_k)$. (Grzelak&Oosterlee,2019)

2.9.3.Options on Zero- coupon bond

The Zero coupon bond is a very important interest rate instrument in the pricing world as it forms most of the time as a numeraire for other underlyings. A European style option is defined by the following equation:

$$V^{zcb} t_0, t\ ) = \ \mathbb{E}^Q[\frac{M(t_0)}{M(T)} \max(\overline{\alpha}\ (P(T, TS) - K)0)|\ f(t_0)\ . \quad (5.10)$$

With $\overline{\alpha}$=1 for a call option and $\overline{\alpha}$= -1 for a put option. The strike price K and $dM(t) = r(t)M(t)dt$ by measure change under the Girasonov theorem from the forward measure T the pricing eqaution under expectation is expressed the following way:

$V^{zcb}(t_0, t) = P(t_0, T)\mathbb{E}^T[\max \overline{\alpha}\,(P(T, TS) - k), 0|\ f(t_0)]$ (Under the Hull-White model).

The pricing equation changes when dealing with an affine short rate model "r(t)" the ZCB becomes an exponential function the pricing equation then becomes:

$V^{zcb}(t_0, T) = P(t_0, T)\mathbb{E}^T\left[\max \overline{\alpha}\,(\,e^{\overline{A_r}(\tau)+\widehat{B_r}(\tau)r(\tau)} - K\right), 0)|\ f(t_0)]$

$=P(t_0, T)e^{\overline{A_r}\,(T)}\mathbb{E}^T[\max(\ \widehat{\alpha}\,(e^{\widehat{B_r}\,(\tau)r(\tau)} - \widehat{K}), 0|\ f(t_0)]$ with $\tau = T_s - T, \widehat{K} = Ke^{-\ \widehat{A_r}(\tau)}$ and r(T) is the short rate under the T forward measure the functions $\overline{A_r}\,(T)$ and $\widehat{B_r}(\tau)$ are defined by : $\overline{A_r}\,(T) = \lambda \int_0^T \theta(Ts - Z)\,\widehat{B}(z)dz + \frac{\eta^2}{4\lambda^3}(\,e^{-2\lambda\tau}(4e^{\lambda T} - 1) + \frac{\eta^2}{2\lambda^2}\tau,$ (5.11)

$$\widehat{B}(T) = \frac{1}{\lambda}(e^{-\lambda\tau} - 1)$$

2.9.4. Caplets and floorlets

Let's consider two specific future time points $T_k - 1 < T_k\ with\, T_k = T_k - T_k - 1$ the $T_k - 1$ is regarded as a caplet or floorlet with the rate K_k and nominal amount N_k considered to be a contract at time T_k the amount:

$V_K^{cpl(call)}(T_k) = T_k N_k \max(l_k(T_k - 1) - K_k, 0)$ (5.12)

$V_k^{fl(put)}(T_k) = T_K N_k \max(K_K - l_k(T_k - 1), 0)$ (5.13)

Caplets and florlets are simply vanilla options, with an accumulated interest rate from time $T_n - 1$. Until T_n. Now let's assume that the libor rate follows a log normal distribution with the following expression: $dl(t; T_k - 1, T_k) - T\,K\,l\,(t; T_k - 1, T_k, dW^K(t)$ if we take into consideration Black's 76 dynamics the value of a caplet is given by:

Caplet $K(t_0) = N_k T_k(t_0, T_k)[l(t_0, T_k - 1, T_k)N(d_1) - K_k N(d_2)$ (5.14)

with:

$$d_1 = \frac{\log\left(\frac{l(t_0; T_k - 1, T_k}{K}\right) + \frac{1}{2}\sigma^2 K(T_k - t_0)}{\sigma_k(\sqrt{T_k - t_0}}$$

$$d_2 = d_1 - \sigma K\sqrt{T_k - t_0}$$

Caplets and floorlets can also be priced under a specific type of multi factor model “the Hull White”. The value of a caplet under the hull White model with a strike price “K” is given by:

$$V^{cpl}(t_0) = NT_k \mathbb{E}^{Q}[\max(l_k(T_k - 1) - K, 0)|f(t_0)$$
$$= NT_k\, P(t_0, T_k)\mathbb{E}^{T_k}[\max(l_k(T_k - 1 - K, 0)|f(t_0)$$

By definition of the libor rate, caplet valuation can be written as: $\frac{V^{cpl}(t_0)}{P(t_0,T_k)} = N\,T\,K\,\mathbb{E}^{T_k}[max\frac{1}{T_k}\left(\frac{1}{P(T_k,T_k}-1\right) - K, 0)|f(t_0)$.

The caplets and floorlets prices under the Hull and White model are obtained after several calibration steps. However the implied volatilites under lambda and eta needs to be monitored regulary (Grzelak&Oosterlee, 2019).

The following example shows the pricing process under the hull and White model:

Lamda and eta are estimated both to be:0.02. we have two maturity times one the zero coupon bond: 4 years and another for the caplet (call option): 8 years. The simulation is under 20 thousands montecarlo paths:

The analytical expression vs the montecarlo of the zero coupon bond give us:

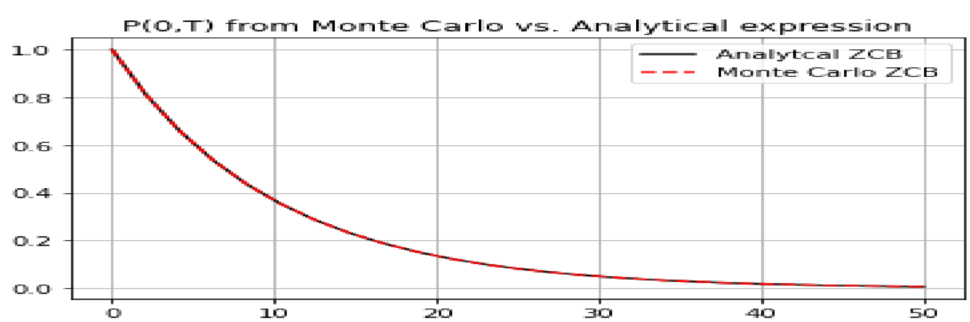


Figure 10: the option under the Zero-coupon bond forward rate

K = np. Linspace(frwd/2.0,3.0*frwd,25) gives the following price:

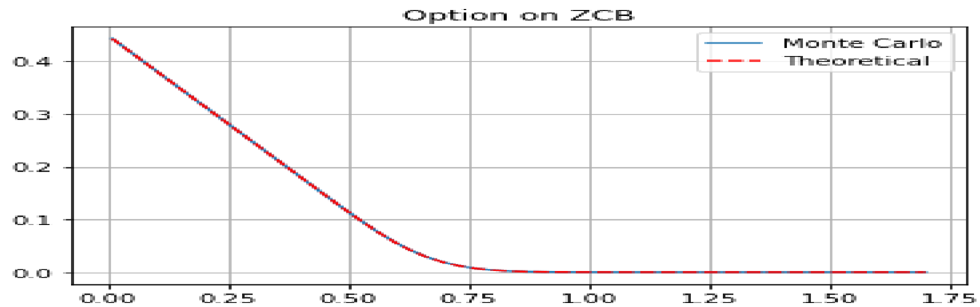


Figure 11: implied volatility generated by lambda parameter under Hull-White

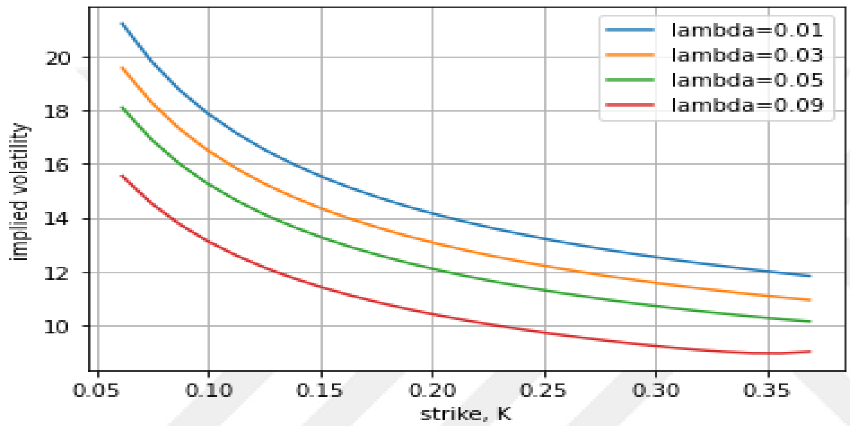
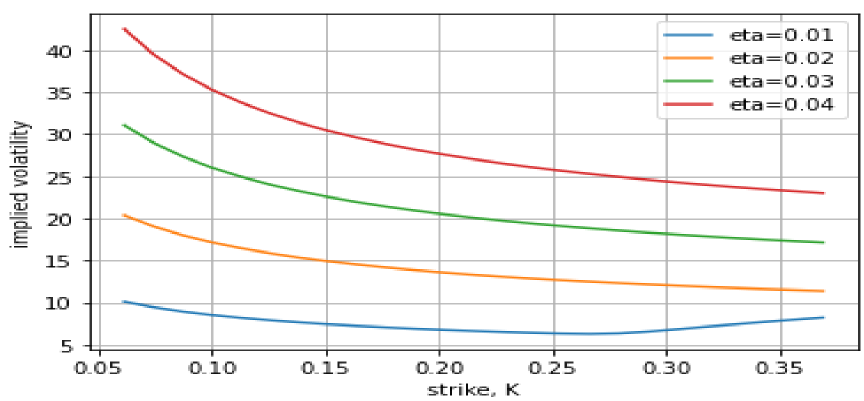


Figure 12: Implied Volatility generated by eta parameter Hull-White



This model is very expensive to compute since it requires firstly datas of the yield curve and secondly the parameters eta and lamda need constantly to be adapted to the market at a monthly frequence, for eta it can be a weekly frequence (Grzelak, 2019).

3. DATA AND METHODOLOGY

This section represents practical methods used for options pricing in the periods of low or negative interest rate in the case of a European type of payoff. In this study, the notoriety of the famous pricing model Black's 76 is preserved. Since the model itsself doesn't recognize negative rates, a shift parameter is assigned to the model in order to enable its capacity of dealing with negative rates.

Price of caplets and floorlets are used for the shifted black model's implementation. The model's calibration is done in Python where numpy for design and matplotlib for maths formulas and graphs design are the first features to be imported for the model's calibration. When calibrating the shifted black model, the first step is to generate the shifted Brownian Motion for a caplet or a floorlet paths. The second step is essentially based on the probability density function where negative value on the axis are recovered by the shift parameter. The last part is for the option price where all the shift parameters follow a same trend to converge on the same price.

The shifted model can also be backed by the exact solution of the montecarlo simulation. Even though its application is not mandatory when pricing, the montecarlo simulation is a good tool for the verification of the exact price of an option.

3.1. Python's codes description

In order to price caplets and floorlets under negative rates, Python was used as a pricing tool. The patterns of prices calibration can be resumed into following steps:

- For the design Numpy needs to be imported, for maths and graphs, Matplotlib and scipy need to be incorporated.
- When calibrating the model, a code has to be assigned in order to distinguish a call option from a put (1 for call and -1 for put)
- The first most important step is to generate geometric Brownian Motion paths. For our shifted model, even in the geometric brownian motion, a shift parameter has to be inserted.
- The following most important step is to make sure the samples have a mean 0 and variance 1.

- Every calibtration has to have standard main calculations. For this study case we consider: the volatility parameters, the interest rate part, the main shift parameter, the time to maturity and the strike price.
- After elaborating the main calculation part, a simulation of the option price, the exact montecarlo solution and the implied volatility should be next step.

3.2.Pricing under negative rates

Pricing under negative rates can be quite challenging since few of the existing models are able to handle negative values. Practictioneers like it's observed on the market some pricing entities (ICAP) choose to add a shift parameter to the existing models like Black's 76 or the SABR model. In this work, the shifted Black will be proposed as the alternative solution to pricing options under negative interest rates. The application case will be done on caplets and floolrets prices simulation with some set of data observed on Thomson Reuters and ICAP.

*** Pricing caplets under negative rates**

Caplets or floorlets can be priced under negative rate evironment by an adaptation of the underlying dynamics of the libor rate. The process consist in adding a shift parameter to the libor rate in the case of a European option we add a shift parameter to the Euribor observed on the market. The shift added to the libor rate is defined the following way:

$$\hat{l}(t) = l_k(t) + \theta_k \quad (6.0)$$

The process of the libor rate is governed by the log normal process under the following Dynamics:

$$d\hat{l}(t; T_k - 1, T_k) = \widehat{\sigma_k}\, \hat{l}(t; T_k - 1, T_k) dW_k^K(t). \quad (6.1)$$

Shifting the process simply means moving the probability density along the X-axis. The shift process requires some sort of the delicacy when choosing a shift parameter. A pricing expert have to make sure that the chosen shift is closer to zero. To make sure of the convrgence, one can always back up the pricing formula with the exact simulation of the montecarlo although it's not necessary all the time.

The pricing formula with the shift parameter of a caplet is given by: $V_k^{cpl}(t_0) = N_k T_k P(t_0, T_k)[\widehat{l_k}\,(t_0) N(d_1) - \widehat{K}_k N(d_2)$ (6.2) with

$$d_1 = \frac{\log\left(\frac{l(t_0; T_k - 1, T_k}{K}\right) + \frac{1}{2}\sigma^2 K(T_k - t_0)}{\sigma_k(\sqrt{T_k - t_0}}$$ and

$$d_2 = d_1 - \sigma K\sqrt{T_k - t_0}$$

Shifting parameters are included in the libor rate with $\widehat{K} = K + \theta_k$ and $\widehat{l_k}(t_0) = l_k(t_0) + \theta_k$ (6.3)

4. EMPIRICAL FINDINGS

For an application example we first simulate an option of a one year maturity with an Euribor of 3M at-0.461 the shift parameter is 0.5, the strike price is 0.445 the log normal volatility is 0.75% we get the following results step by step

- The shifted brownian motion:

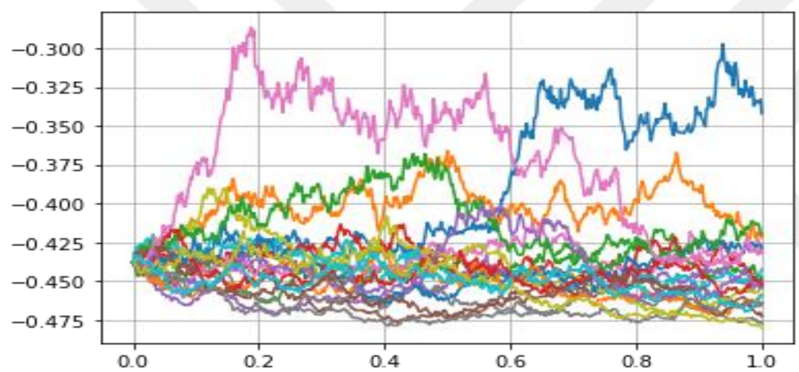


Figure 13: caplets paths under the Brownian motion

- The shifted log normal density:

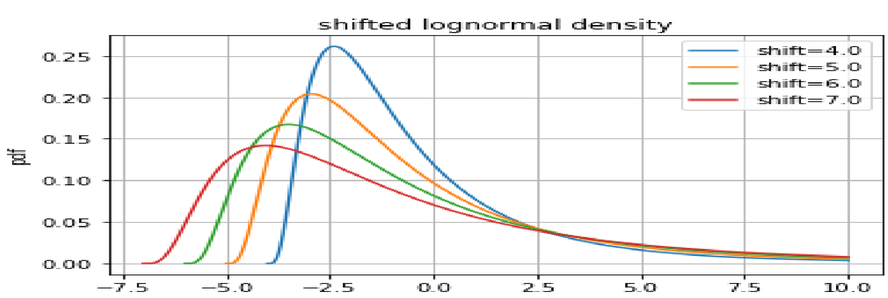


Figure 14: Recovering negative values with the shift parameter

- The real option price

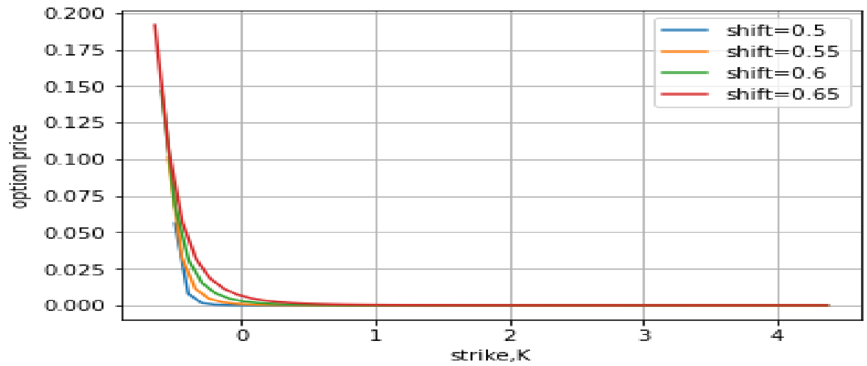


Figure 15: real options prices

The Montecarlo simulation can helps to verify the real price of an option by its exact solution. For the case of this study, caplets and floorlets are going to be tested under different dynamics (volatility, shift parameter, different maturity date,). The real option price will exactly be same as the price given by the exact simulation of the Montecarlo simulation.

From ICAP we get the following datas for a caplet:

T = 3.0

Sigma = 0.35

L0 =-0.045

Shift = 0.1

K (ATM) = [0.98]

CP = OptionType.CALL. Since we are using the montecarlo simulation, let's simulate under 500 number of steps and 10000 number of paths.

1. The first step is to generate geometric Brownian Motion paths

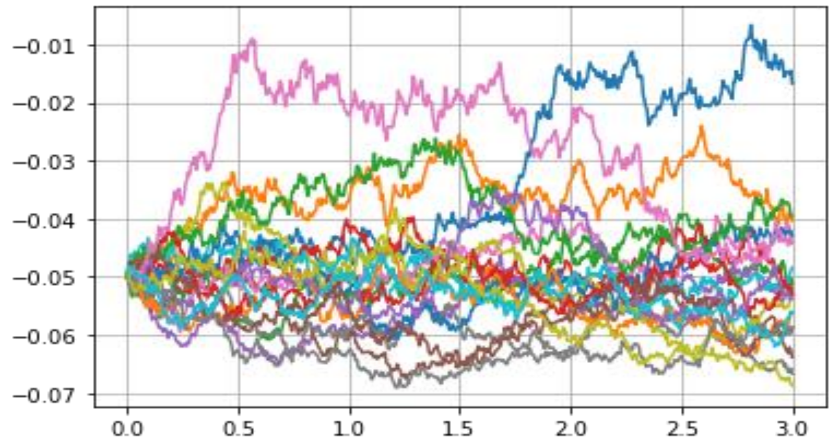


Figure 16: Shifted Brownian Motion

1. The second step we generate the shifted lognormal density.

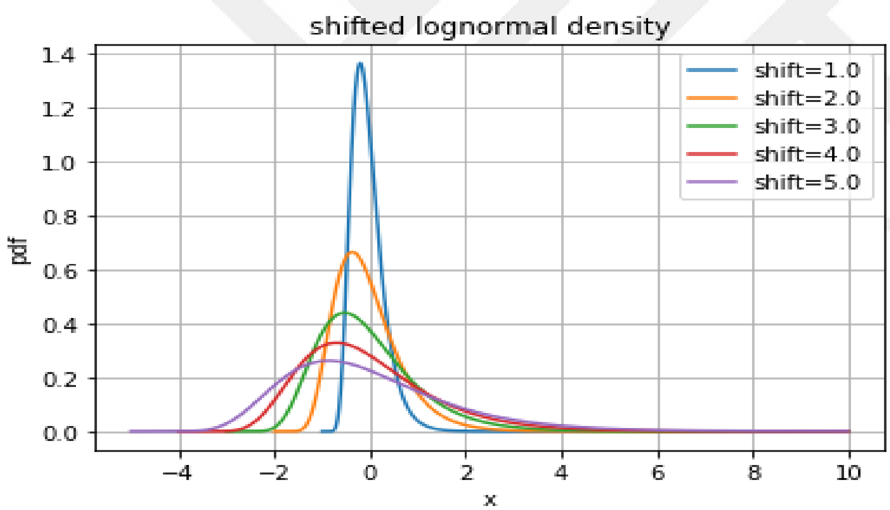


Figure 17: shifted log normal density

2. The third step we calibrate the exact solution of the montecarlo simulation to generate the option price.

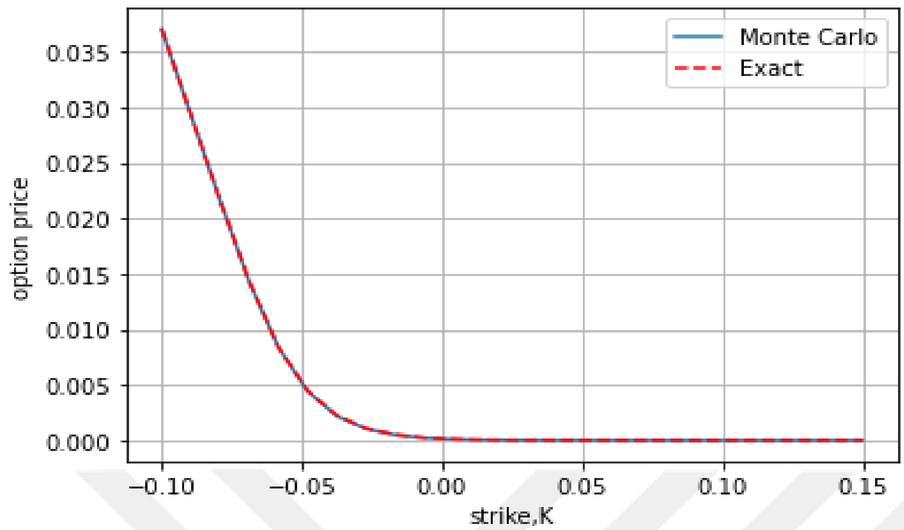


Figure 18: the exact solution and the Monte Carlo simulation

3. The last step is to generate the option's price. In this case the option price has to be simular with the price generated by the montecarlo simulation since but since in the last one has a standard error, the price might slightly differ.

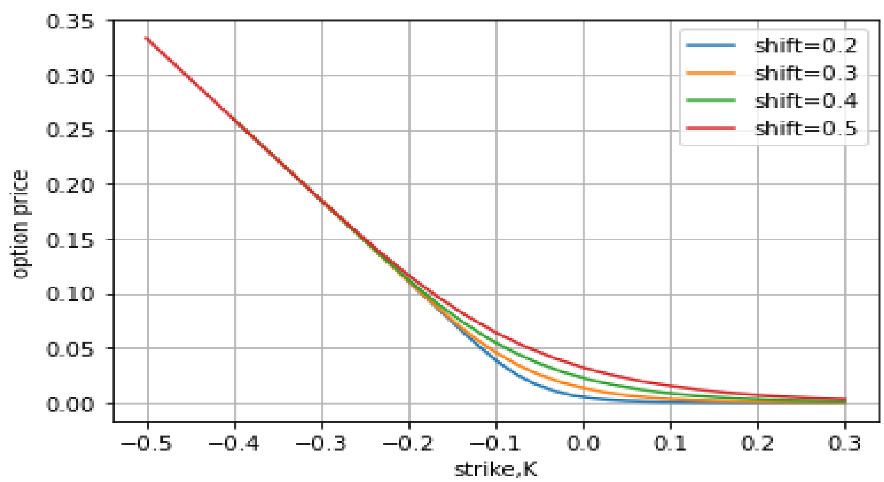


Figure 19: the real option price

Shift parameters determines the lowest level of negative interest rates. However due to constant change of the Euribor, the shift parameter has to be adjusted frequently. The challenging part about the shifted Black model is that it has to be manipulated with precaution. The shift parameter must not be very bigger than the actual libor rate. It has to be the closest possible. Due to that argument of the right shift parameter to use, the market convention often provides the informtions on the right shift parameter to use.

The shift parameter is always associated with the time to maturity of the underlying. The shifted black model is slightly different from the classic Black Scholes model. In order to prevent negative rates to impact the strike price the shift parameter is also added to the strike parameter. The strike price becomes $\hat{K} = K + \theta_k$ under the new libor $\hat{l}_k(t_0) = l_k(t_0) + \theta_k$

The only problem with the shifted black model, the implied volatility has to be found which may lead some practitioner to prefer the shifted SABR model over the shifted Black model. In practice, when calibrating the shifted Black model, the implied volatility has to be calculated. If a wrong volatility value is used, the option price will be erroneous.

5. CONCLUSION

Options are in general a good instrument for hedging. As it was seen in the historic review of options trading, the mathematician Thales of Miletus from the ancient Greece profited from the olive harvest to create the first call option in the history. Options are regarded as one of the most complex subjects in finance as it was displayed in some theories of this study. The dynamics behind Options pricing involve a lot of parameters. The first most important concept is to understand the logic behind an asset price. The Brownian Motion illustrates an asset's price movement and the uncertainty or the volatility of an asset is defined by the volatility parameter $\boldsymbol{\sigma}$. In order to calibrate an asset's price probable movements the following stochastic equation is used: $\frac{dS(t)}{S(t)} = \mu dt + \sigma dW^p(t)$. The first term defines the drift term and the Second term is the volatility parameter associated with the weinar process.

Determining the stochastic differential equation of an underlying is the first important step in a pricing approach. By using the feynamn-kac formula or the Ito's derivation, a preferential diferential equantion that gives the real price of an option is found. SDE's and PDE's can be under different measures $(\boldsymbol{Q}, \boldsymbol{P}, ..)$. The pricing equation should however be under the risk neutral measure $(\boldsymbol{Q})$. Under the martingale approach to pricing, the stochastic differential equation with a drift parameter μ relies on statisitical number from the past to predict a future stock price (grzelach,2019). For better forecasts, market dyanamics of pricing have to be considered. Under the martingale the stochastic equation of an asset has to be driftless.

In an asset's historic review, some unnormal prices distribution can be observed. For example in 2008, SP500 registered record low of -46%. On the graph it's a huge gap compared to previous years. The jump diffusion like specified in this studty, analyses unnormal jumps in stocks in order to predict future stock paths. The Jump diffusion model is defined under the following equation: $X(t) = \mu dt + \sigma dW^p(t) + \jmath dX_p(t)$

The momentum between unnormal jumps start and the moment after the jumps is defined by the parameter $\jmath$ positioned in the front of the Brownian Motion $dX_p(t)$.

Negative rates like stated in the introductive part of this study, are somehow a new concept that was established in order to boost the economy that was recovering from a recession. Denmark was the first to implement negative rate policy in 2012 followed by the European central bank in 2016.

The implementation of negative rates can have both positive and negative impacts. On one hand borrowing becomes cheaper and there's a lot of cash inflows in the economy. On the other hand rate cuts below zero, implies for commercial bank that there's no longer cash reserves to hold on since it can result into a money- loss activity. With negative rates, comes the ptoblem of reviewing pricing models like it's the case for derivatives where classic models like the Black Scholes model where built under the assumption that rates will never become negative.

The derivative world has a lot of pricing models each with different features. In this study, simple European type of options were analysed. The less models are complex, the more they are likely to be used on the market. When it's question of speed there's no doubt fast fourier transform models give accurate results. For large and complex numbers the Montecarlo simulation is often preferred.

Choosing a pricing model depends on the type of instrument one wants to price. From the model analysed in this study, the Hull and White models, the SABR model and Black Scholes model seem to be the models with huge stability that can handle any type of derivative. The SABR model has a good performance when it comes to the negative rate environement because it can also incorporate a shift parameter in its features. The only problem with that model is it's complexity but as advised by deloitte (october 2016), the free boundary SABR by Antonov et al seems less complex when dealing with negative rates. The Hull and White model is gaussian distribution type of model. Which implies that there is a probability of rates being negatives. The Gaussian properties of Hull-White 1 factor model is:

$\boldsymbol{p(r(t) \leq x)} = \phi(\frac{\boldsymbol{x-\mathbb{E}(r(t))}}{\boldsymbol{Var(r(t))}})$. The probability of rates becoming negative therefore becomes:

$\boldsymbol{p(r(t) \leq 0} = \phi(-\frac{\boldsymbol{r(s)e^{-a(t-s)}+\alpha(t)-\alpha(s)e^{-a(t-s)}}}{\sqrt{\boldsymbol{\frac{\alpha}{2a}(1-e^{-2a(t-s)})}}})$. The Hull and White model is a good model for the derivative pricing in both and positive and negative environement. Its calibration is not fast and is often expensive to use hence its rare application on the market. For this sudy the shifted Black-Scoles model was used to price options under low or negative rates environement. Even though choosing the shift parameter can be challenging, the model is handy and can be very well applied on today's market. The following is a table that describes some reviewed models in this study, their advantages and their inconvenients.

Models	Advantages	Inconvenients
Shifted Black-Scholes	Very handy and easy to use. Provide accurate results when backed with the monte carlo simulation.	The determination of the right shift parameter can be challenging. With long maturities negative the shift parameter can't easily adapt.
Fast fourier transform	Fast computtion and accurate results for simple payoffs.	Only applied on simple payoffs and need constantly to be updated. Tend to give a negative value when rates are negative.
Heston model	Advised for multiple strikes or for options with unusual payoffs like a cash or nothing option.	The cox ingersoll doesn't have allow negative values of negative interest rates.
SABR	Good model for stochastic volatility and European type of options	Hard computation and derivation
Hull White model	The model can be a multi-factor instead of a single factor. Can be handy for pricing caplets and floorlets. Thanks to its strong features, it can be used to design a yield curve	Expensive to compute and the paramaters of eta and lambda need to be constat-ntly changed and adapted to the market.

Monte carlo model for option pricing	Can handle long numbers and be used for sophisticated payoffs	The more large numbers are inserted in the model the more the standard errors grow

As suggestion, the shifted Black model, can be a good fit for the actual negative rate environement. Inserting a shift parameter solve the problem of negative rates and provides accurate prices. Future works will this time be on a model that can incorporate yield curve datas in the options pricing process by first elaborating the option price under zero coupon- bond. The second future study will be oriented on pricing with the shifted SABR model.

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